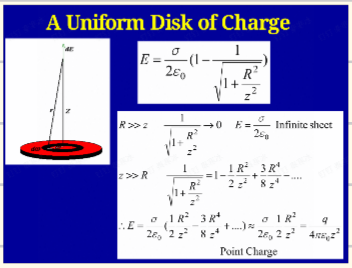
9. Coulomb's Law

$e = 1.602\times10^{-19}C$ $q = ne$ electric charge is quantized

$F_{12} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1q_2}{r_{12}^2}\hat{e}_{12}$ 1C = 1A·s 1C等于电流为1A时，1s内流过导线中任一截面的电量.

·叠加：矢量和 $F = \sum_{i=2}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_1q_i}{r_{1i}^2}\hat{e}_{1i}$


A Uniform Disk of Charge

$$E = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right)$$

$R >> z$ $\frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\to 0$ $E = \frac{\sigma}{2\varepsilon_0}$ Infinite sheet

$z >> R$ $\frac{1}{\sqrt{1+\frac{R^2}{z^2}}} = 1-\frac{1}{2}\frac{R^2}{z^2}+\frac{3}{8}\frac{R^4}{z^4}-\ldots$

$\therefore E = \frac{\sigma}{2\varepsilon_0}\left(\frac{1}{2}\frac{R^2}{z^2}-\frac{3}{8}\frac{R^4}{z^4}+\ldots\right)\approx\frac{\sigma}{2\varepsilon_0}\frac{1}{2}\frac{R^2}{z^2} = \frac{q}{4\pi\varepsilon_0 z^2}$

Point Charge

·continuous charge distributions

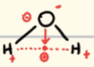
i) Thin charged rods : λ linear charge density $dq = \lambda dx$ $\lambda = \frac{q}{L}$

ii) Surface of the carrier bead : σ surface charge density $dq = \sigma dA$ $\sigma = \frac{q}{A}$

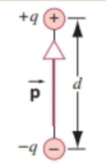
iii). Volume of a 3D object : ρ volume charge density $dq = \rho dV$ $\rho = \frac{q}{V}$

ps. 电场以光速传播

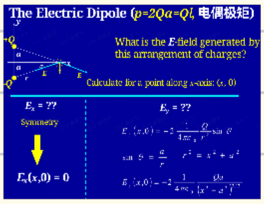
A dipole in an electric field : Two Opposite Charges : Dipole (电偶极矩) eg. H_2O



正负电荷中心不重合.



dipole moment vector $\vec{p}$: $p = qd$ (negative → positive)


The Electric Dipole ($p=2Qa=Ql$, 电偶极矩)

What is the E-field generated by this arrangement of charges?

Calculate for a point along x-axis: $(x, 0)$

$E_x = ??$

Symmetry

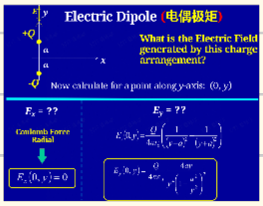
$E_x(x,0) = 0$

$E_y = ??$

$E_y(x,0) = -2\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\sin\theta$

$\sin\theta = \frac{a}{r}$ $r^2 = x^2 + a^2$

$E_y(x,0) = -2\frac{1}{4\pi\varepsilon_0}\frac{Qa}{(x^2+a^2)^{3/2}}$


Electric Dipole (电偶极矩)

What is the Electric Field generated by this charge arrangement?

Now calculate for a point along y-axis: $(0, y)$

$E_x = ??$

Coulomb Force Radial

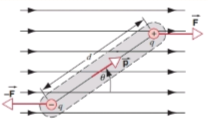
$E_x(0,y) = 0$

$E_y = ??$

$E_y(0,y) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{(y-a)^2}-\frac{1}{(y+a)^2}\right)$

$E_y(0,y) = \frac{Q}{4\pi\varepsilon_0}\frac{4ay}{y^4\left(1-\frac{a^2}{y^2}\right)^2}$

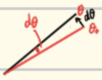
⇒ $\tau = \vec{r}\times\vec{F}$



$F = qE$ (opposite directions) torque on each charge : $\tau = Fr_{\perp}$ net torque : $\tau = F\frac{d}{2}\sin\theta + F\frac{d}{2}\sin\theta = Fd\sin\theta$ (perpendicular to the plane of the page and into the page). 垂直的 $\tau = (qE)d\sin\theta = (qd)E\sin\theta = pE\sin\theta$ ⇒ $\vec{\tau} = \vec{p}\times\vec{E}$

Work from an initial angle θ_0 to a final angle θ is : $W = \int dW = \int_{\theta_0}^{\theta}\vec{\tau}\cdot d\vec{\theta} = \int_{\theta_0}^{\theta}-\tau d\theta$ ★ $= -\int_{\theta_0}^{\theta}pE\sin\theta d\theta = pE(\cos\theta-\cos\theta_0)$

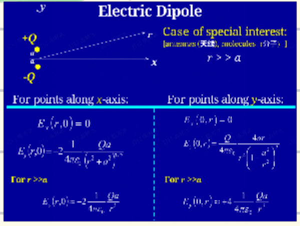
力矩作功: $dW = \vec{\tau}\cdot d\vec{\theta}$



magnitude: $\tau = d\cdot F\cdot\sin\theta$ ⇒ $\underset{dW=}{\vec{\tau}\cdot d\vec{\theta}} = -\underset{x}{\underline{d\cdot d\theta}}\cdot\underset{F}{\underline{F\sin\theta}}$

⇒ $\Delta U = -W = U(\theta)-U(\theta_0) = -pE(\cos\theta-\cos\theta_0)$ define: $\theta_0 = 90°$ $U(90°) = 0$

$U(\theta) = -pE\cos\theta = -\vec{p}\cdot\vec{E}$


Electric Dipole

Case of special interest:
[antennas (天线), molecules (分子)]

$r >> a$

For points along x-axis:

$E_x(r,0) = 0$

$E_y(r,0) = -2\frac{1}{4\pi\varepsilon_0}\frac{Qa}{(r^2+a^2)^{3/2}}$

For $r >> a$

$E_y(r,0) = -2\frac{1}{4\pi\varepsilon_0}\frac{Qa}{r^3}$

For points along y-axis:

$E_x(0,r) = 0$

$E_y(0,r) = \frac{Q}{4\pi\varepsilon_0}\frac{4ar}{r^4\left(1-\frac{a^2}{r^2}\right)^2}$

For $r >> a$

$E_y(0,r) = +4\frac{1}{4\pi\varepsilon_0}\frac{Qa}{r^3}$

x-axis : $E = \frac{1}{4\pi\varepsilon_0}\frac{p}{(x^2+a^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0 x^3}\cdot\frac{p}{(1+(a/x)^2)^{3/2}}$

⇒泰勒 : $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$ ⇒ $E = \frac{p}{4\pi\varepsilon_0 x^3}[1+(-\frac{3}{2})\cdot(\frac{a}{x})^2+\cdots]$

★解释: $d\vec{\theta}$ 方向也是右手定则 ⇒

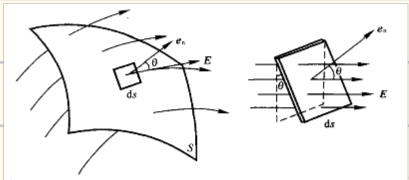
垂直纸面向外为 $d\vec{\theta}$ 方向

9 电场线、电通量

通量(Flux) $\Rightarrow$ 流量 $\Phi = \oiint \vec{v} \cdot d\vec{A}$

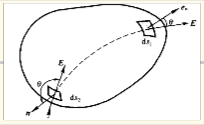
垂直于场强方向的面积元 dS 上，通过的电场线数 dN（电场线数密度）正比于该点场强 E 的大小 ： $E = \frac{dN}{dS}$

电通量：电场中任取面积元 dS 可视为平面，dS 所在处的场强 E 也可认为是均匀的。用面积元的法向单位矢量 $\hat{e}_n$，规定其正方向，面积元表示为矢量 $d\vec{S} = dS\hat{e}_n$. $d\vec{S}$ 与所在处的场强 E 的方向成 θ 角，则定义 $d\Phi_e = EdS\cos\theta = \vec{E} \cdot d\vec{S}$ 为通过面积元 dS 的电通量. 通过 dS 的电通量即为通过该面积的电场线总数.



- 叠加得到通过曲面 S 的总电通量，即 $\Phi_e = \int d\Phi_e = \int_S \vec{E} \cdot d\vec{S} = \int_S E\cos\theta dS$ （Φ_e 标量，有正负，$\theta < \frac{\pi}{2}$ 正，$\theta > \frac{\pi}{2}$ 负

.通过闭合曲面的通量可以写成：$\Phi_e = \oint_S E\cos\theta dS = \oint_S \vec{E} \cdot d\vec{S}$



通常规定垂直曲面向外为法线正方向

PS. 积分 $\oint_S$ 写成 $\oiint$

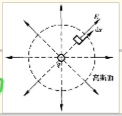
9. 高斯定理及其应用

- 通过任意闭合曲面的电通量等于该曲面所包围的所有电量的代数和除以 ε_0. 表达式为：$\Phi_e = \oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0}\sum q_i$ （闭合曲面 S 称为高斯面）

定义向外为正

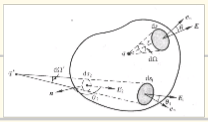
Proof. 点电荷

(1) 球形闭合面



$\Phi_e = \oint_S \vec{E} \cdot d\vec{S} = \oint \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} dS = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$

(2). 任意闭合面



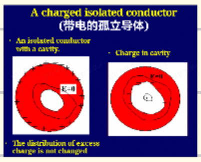
dS 边缘上各点向点电荷所在处引直线，形成锥体，其顶角为立体的，则该顶点为面积元 dS 对点电荷所张的立体角

立体角用符号 $d\Omega = \frac{dS\cos\theta}{r^2}$ （代数量） 闭合曲面对面内 q 所张的立体角总值为 4π

内部. $\Phi_e = \oint_S d\Phi_e = \oint_S \frac{q}{4\pi\varepsilon_0 r^2} dS\cos\theta = \frac{q}{4\pi\varepsilon_0}\oint_S d\Omega = \frac{q}{\varepsilon_0}$

外部：θ_1 为锐角，θ_2 为钝角，两者立体角数值均为 $d\Omega'$，符号相反 $d\Phi_{e_1} = \vec{E}_1 \cdot d\vec{S}_1 = \frac{q'}{4\pi\varepsilon_0} d\Omega'$ $d\Phi_{e_2} = \vec{E}_2 \cdot d\vec{S}_2 = -\frac{q'}{4\pi\varepsilon_0} d\Omega'$

$\Rightarrow$ 闭合曲面外电荷对闭合曲面的电通量无贡献


A charged isolated conductor
(带电的孤立导体)

- An isolated conductor with a cavity.
- Charge in cavity

The distribution of excess charge is not changed

(3). 闭合曲面包围 n 个点电荷 $\Phi_e = \oint_S \vec{E} \cdot d\vec{S} = \oint_S (\vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N) \cdot d\vec{S} = \sum_{i=1}^{n} \oint_S \vec{E}_i \cdot d\vec{S} + \sum_{i=n+1}^{N} \oint_S \vec{E}_i \cdot d\vec{S} = \sum_{i=1}^{n} \frac{q_i}{\varepsilon_0}$

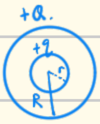
若场源电荷连续：$\oiint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0}\iiint \rho dV$ ρ 为电荷密度，dV 为体积元.

应用：(1). 均匀带电球壳：1). 球壳外：$\Phi_e = \oiint \vec{E} \cdot d\vec{S} = E\oiint dS = E \cdot (4\pi r'^2) = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r'^2}$ 矢量式：$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{e}_r$

ps. 如果是带电导体，那么电荷分布于其外表面，内部场强为零

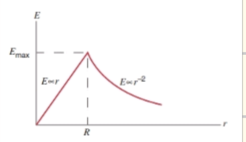
2). 球壳内：高斯面包围空间无电荷 $\Phi_e = \oiint \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0} = 0 \Rightarrow 4\pi r^2 \cdot E = 0 \Rightarrow E = 0$

(2). 均匀（绝缘）带电球体：内部：

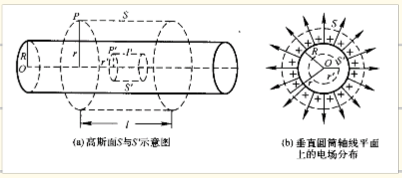


总电荷为 +Q $\therefore \rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$ 内部 $V' = \frac{4}{3}\pi r^3$ $q = \rho \cdot V' = \frac{Qr^3}{R^3}$

$\therefore \Phi_e = \oiint \vec{E} \cdot d\vec{S} = E \cdot \oiint dS = E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{Qr}{4\pi\varepsilon_0 R^3}$ $\therefore E = \begin{cases} \frac{Qr}{4\pi\varepsilon_0 R^3}, & r < R \\ \frac{Q}{4\pi\varepsilon_0 r^2}, & r > R. \end{cases}$



(3). "无限长"均匀带电圆柱面的场强分布. 设带电圆柱面半径为 R，电荷线密度为 λ



(a) 高斯面 S 与 S' 示意图

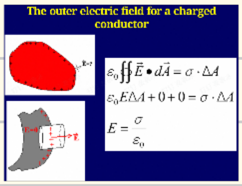
(b) 垂直圆筒轴线平面上的电场分布

(i). 圆柱面外的场强分布：半径为 r (r > R)，长为 l 的闭合圆柱面 S 为高斯面

$\therefore \Phi_e = \oiint \vec{E} \cdot d\vec{S} = \oiint \vec{E} \cdot d\vec{S}_{侧} + \oiint \vec{E} \cdot d\vec{S}_{底} = \oiint \vec{E} \cdot d\vec{S}_{侧} = E \cdot 2\pi r l$ 包围的总电量为 λl $\Rightarrow$ 高斯定理：$E \cdot 2\pi r l = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$

(ii) 圆柱面内的场强：包围无电荷 $\Phi_e = \frac{0}{\varepsilon_0} = 0 = \oiint \vec{E} \cdot d\vec{S} \Rightarrow E = 0$

(4). "无限大"均匀带电平面的场强分布：设电荷面密度为 σ


The outer electric field for a charged conductor

$\varepsilon_0 \oiint \vec{E} \bullet d\vec{A} = \sigma \cdot \Delta A$

$\varepsilon_0 E\Delta A + 0 + 0 = \sigma \cdot \Delta A$

$E = \frac{\sigma}{\varepsilon_0}$

求一点的 E，取极小圆柱高斯面

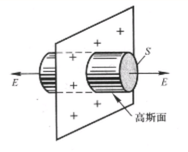


图 9.14 "无限大"均匀带电平面的电场

$\Phi_e = \oiint \vec{E} \cdot d\vec{S} = \oiint \vec{E} \cdot d\vec{S}_{底} = ES + ES = 2ES = \frac{S \cdot \sigma}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$

匀强电场

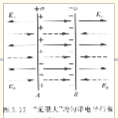
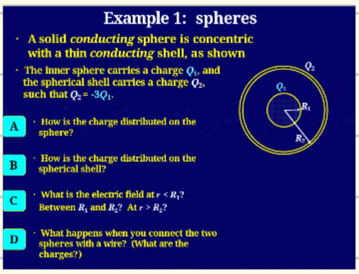


图 9.15 "无限大"均匀带电平行板的电场

平面间 $E = E_A + E_B = \frac{\sigma}{\varepsilon_0}$ ，外侧场强：$E = E_A - E_B = 0$

eg.



Example 1: spheres

- A solid *conducting* sphere is concentric with a thin *conducting* shell, as shown
- The inner sphere carries a charge Q_1, and the spherical shell carries a charge Q_2, such that $Q_2 = -3Q_1$.

A · How is the charge distributed on the sphere?

B · How is the charge distributed on the spherical shell?

C · What is the electric field at $r < R_1$? Between R_1 and R_2? At $r > R_2$?

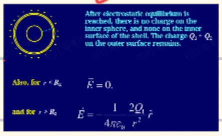
D · What happens when you connect the two spheres with a wire? (What are the charges?)

(A). 由于高斯定理，导体内无电荷

(B). 内表面：$\sigma_{inner} = -\frac{Q_1}{4\pi R_2^2}$ ，外表面：$\sigma_{outer} = \frac{Q_2+Q_1}{4\pi R_2^2} = \frac{-2Q_1}{4\pi R_2^2}$

(C). $r<R$: $\vec{E}=0$ $\quad R_1<r<R_2$: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2}\hat{r}$ $\quad r>R_2$: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1+Q_2}{r^2}\hat{r} = -\frac{1}{4\pi\varepsilon_0}\frac{2Q_1}{r^2}\hat{r}$

(D)



After electrostatic equilibrium is reached, there is no charge on the inner sphere, and none on the inner surface of the shell. The charge Q_1+Q_2 on the outer surface remains.

Also, for $r < R_2$ $\quad \vec{E} = 0.$

and for $r > R_2$ $\quad \vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{2Q_1}{r^2}\hat{r}$

9. 静电场的环路定理.

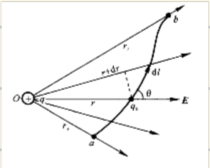


图 9.16 计算电场力做功用图

受力 $\vec{F} = q_0\vec{E}$ 微元，当移动 dl 时，所做的元功：$dW = \vec{F}\cdot d\vec{l} = q_0\vec{E}\cdot d\vec{l} = q_0 E\cos\theta\, dl = q_0 E\, dr = q_0 \cdot \frac{q}{4\pi\varepsilon_0 r^2} dr$

$dl\cos\theta = dr$ $\quad E = \frac{q}{4\pi\varepsilon_0 r^2}$

$\therefore W = \int dW = \int_{r_a}^{r_b}\frac{q_0 q}{4\pi\varepsilon_0 r^2}dr = \frac{q_0 q}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}dr = \frac{q_0 q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$

试验电荷在静止点电荷电场中移动时，电场力做的功仅与试验电荷电量的大小及其起点和终点位置有关，与路径无关

⇒拓展到任何静电场依旧成立 $\qquad$ 静电场是保守场 ⇒ 引入势的概念

·环路定理：$\int_{acb} q_0\vec{E}\cdot d\vec{l} = \int_{adb} q_0\vec{E}\cdot d\vec{l}$ $\quad \int_{acb} q_0\vec{E}\cdot d\vec{l} = -\int_{bda} q_0\vec{E}\cdot d\vec{l}$ $\quad \Rightarrow \oint \vec{E}\cdot d\vec{l} = 0$ 在静电场中，电场强度沿任意闭合回路的线积分恒等于0.

$\nabla\times\vec{E}=0$

∇：哈密顿算子：$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ $\qquad \nabla\times\vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$

Stokes 公式：$\oint_\Gamma P dx + Q dy + R dz = \iint_\Sigma \begin{vmatrix} dy\,dz & dz\,dx & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

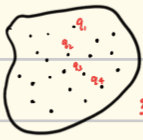
$\oint \vec{E}\cdot d\vec{l} = \iint_S (\nabla\times\vec{E})\cdot d\vec{S} = 0 \Rightarrow \nabla\times\vec{E} = 0$

9. 电势能 & 电势

$U_a - U_b = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$ $\quad r_a\to\infty$ $\quad \therefore U(r) = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$ 结合能

试验电荷 q_0 在静电场中任一给定位置时，它具有一定的电势能，若将 q_0 从 a点移至 b点，其间电场力做功等于电荷静电势能增量的负值：$W_{ab} = -(U_b - U_a) = -\Delta U$

确定电势能的绝对值 ⇒ 选取电势能为0的参考点，无限远处电势能为0 $\quad U_\infty = 0 \Rightarrow U_p = W_{p\infty} = \int_p^\infty q_0\vec{E}\cdot d\vec{l}$ $\qquad q_0$ 在电场中P的电势能 U_p 数值上



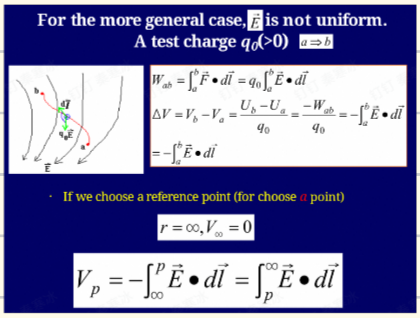
结合能

$U = U_{12} + U_{13} + U_{14} + \cdots + U_{1N}$
$+ U_{23} + U_{24} + \cdots + U_{2N}$
$+ U_{N-1\,N}$

$\Rightarrow U = \sum_{i\neq j}\frac{1}{2}U_{ij}$

$\Rightarrow W_{AB} = -\Delta U = \int_A^B \vec{F}\cdot d\vec{l} \Rightarrow \Delta U = U_b - U_a = -\int_A^B q_0\vec{E}\cdot d\vec{l} \Rightarrow V_b - V_a = -\int_A^B \vec{E}\cdot d\vec{l}$

电势：$V = \frac{U}{q_0} = \frac{q}{4\pi\varepsilon_0 r}$ $\qquad W_{AB} = -\int_A^B \vec{F}_{elec}\cdot d\vec{l} = -\int_A^B q_0\vec{E}\cdot d\vec{l} \Rightarrow V_B - V_A = \frac{W_{AB}}{q} = -\int_A^B \vec{E}\cdot d\vec{l}$ 电势差.



For the more general case, $\vec{E}$ is not uniform.
A test charge $q_0(>0)$ $a \Rightarrow b$

$W_{ab} = \int_a^b \vec{F}\bullet d\vec{l} = q_0\int_a^b \vec{E}\bullet d\vec{l}$

$\Delta V = V_b - V_a = \frac{U_b - U_a}{q_0} = \frac{-W_{ab}}{q_0} = -\int_a^b \vec{E}\bullet d\vec{l}$

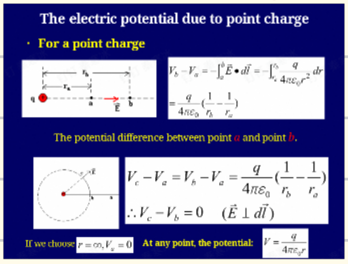
$= -\int_a^b \vec{E}\bullet d\vec{l}$

- If we choose a reference point (for choose *a* point)

$r = \infty, V_\infty = 0$

$V_p = -\int_\infty^p \vec{E}\bullet d\vec{l} = \int_p^\infty \vec{E}\bullet d\vec{l}$

$r\to\infty$ $V_b = 0$ $\therefore 0 - V_a = -\int_A^B \vec{E}\cdot d\vec{l} \Rightarrow V_a = \int_A^\infty \vec{E}\cdot d\vec{l}$



The electric potential due to point charge

- For a point charge

$V_b - V_a = -\int_a^b \vec{E}\bullet d\vec{l} = -\int_{r_a}^{r_b}\frac{q}{4\pi\varepsilon_0 r^2}dr$

$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right)$

The potential difference between point *a* and point *b*.

$V_c - V_a = V_b - V_a = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right)$

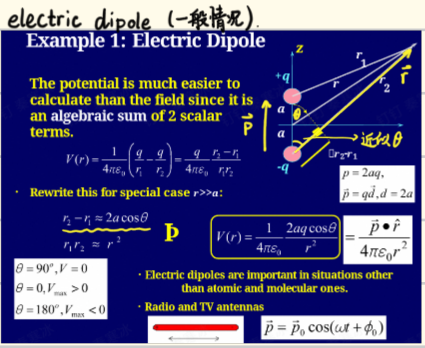
$\therefore V_c - V_b = 0 \quad (\vec{E}\perp d\vec{l})$

If we choose $r = \infty, V_\infty = 0$ At any point, the potential: $V = \frac{q}{4\pi\varepsilon_0 r}$

$$\boxed{V_a - V_b = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)}$$

$$\boxed{V_p = \int_p^\infty \vec{E}\cdot d\vec{l}}$$

electric dipole (一般情况).



Example 1: Electric Dipole

The potential is much easier to calculate than the field since it is an algebraic sum of 2 scalar terms.

$$V(r)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1}-\frac{q}{r_2}\right)=\frac{q}{4\pi\varepsilon_0}\frac{r_2-r_1}{r_1r_2}$$

- Rewrite this for special case $r>>a$:

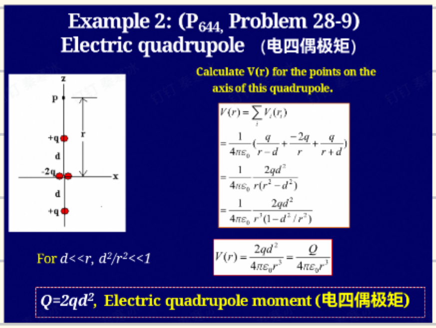
$$\underbrace{r_2-r_1\approx 2a\cos\theta}_{} \quad r_1r_2\approx r^2 \quad Þ \quad \boxed{V(r)=\frac{1}{4\pi\varepsilon_0}\frac{2aq\cos\theta}{r^2}}=\frac{\vec{p}\bullet\hat{r}}{4\pi\varepsilon_0 r^2}$$

$p=2aq,$
$\vec{p}=q\vec{d},d=2a$

$\theta=90^o,V=0$
$\theta=0,V_{\max}>0$
$\theta=180^o,V_{\max}<0$

- Electric dipoles are important in situations other than atomic and molecular ones.
- Radio and TV antennas

$\vec{p}=\vec{p}_0\cos(\omega t+\phi_0)$



Example 2: (P$_{644}$, Problem 28-9) Electric quadrupole （电四偶极矩）

Calculate V(r) for the points on the axis of this quadrupole.

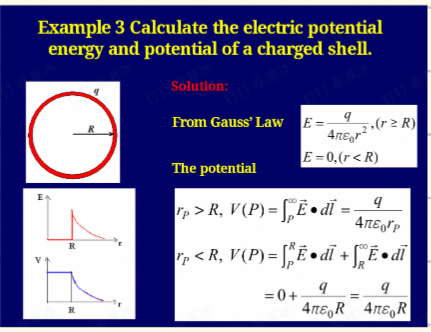
$$V(r)=\sum_i V_i(r_i)$$
$$=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r-d}+\frac{-2q}{r}+\frac{q}{r+d}\right)$$
$$=\frac{1}{4\pi\varepsilon_0}\frac{2qd^2}{r(r^2-d^2)}$$
$$=\frac{1}{4\pi\varepsilon_0}\frac{2qd^2}{r^3(1-d^2/r^2)}$$

For $d<<r$, $d^2/r^2<<1$

$$V(r)=\frac{2qd^2}{4\pi\varepsilon_0 r^3}=\frac{Q}{4\pi\varepsilon_0 r^3}$$

$Q=2qd^2$, Electric quadrupole moment (电四偶极矩)

有电场的地方就有能量!



Example 3 Calculate the electric potential energy and potential of a charged shell.

Solution:

From Gauss' Law

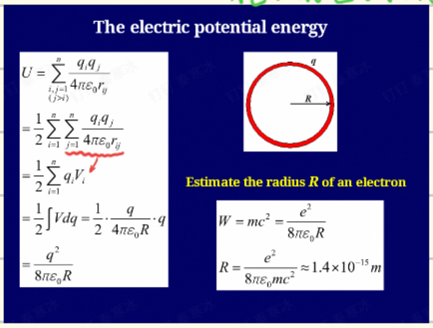
$$E=\frac{q}{4\pi\varepsilon_0 r^2},(r\ge R)$$
$$E=0,(r<R)$$

The potential

$$r_P>R,\ V(P)=\int_P^\infty\vec{E}\bullet d\vec{l}=\frac{q}{4\pi\varepsilon_0 r_P}$$
$$r_P<R,\ V(P)=\int_P^R\vec{E}\bullet d\vec{l}+\int_R^\infty\vec{E}\bullet d\vec{l}$$
$$=0+\frac{q}{4\pi\varepsilon_0 R}=\frac{q}{4\pi\varepsilon_0 R}$$

球壳内部的电势

恒定为 $\frac{q}{4\pi\varepsilon_0 R}$.

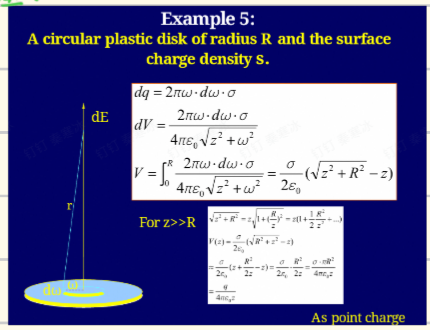


The electric potential energy

$$U=\sum_{\substack{i,j=1\\(j>i)}}^{n}\frac{q_iq_j}{4\pi\varepsilon_0 r_{ij}}$$
$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_iq_j}{4\pi\varepsilon_0 r_{ij}}$$
$$=\frac{1}{2}\sum_{i=1}^{n}q_iV_i$$
$$=\frac{1}{2}\int Vdq=\frac{1}{2}\cdot\frac{q}{4\pi\varepsilon_0 R}\cdot q$$
$$=\frac{q^2}{8\pi\varepsilon_0 R}$$

Estimate the radius R of an electron

$$W=mc^2=\frac{e^2}{8\pi\varepsilon_0 R}$$
$$R=\frac{e^2}{8\pi\varepsilon_0 mc^2}\approx1.4\times10^{-15}m$$



Example 5:

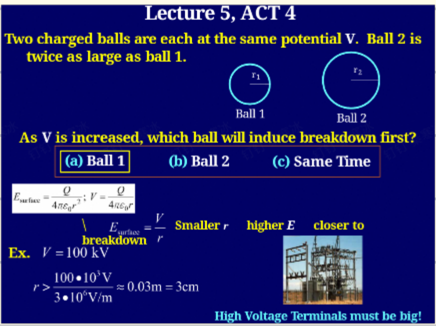
A circular plastic disk of radius R and the surface charge density s.

$$dq=2\pi\omega\cdot d\omega\cdot\sigma$$
$$dV=\frac{2\pi\omega\cdot d\omega\cdot\sigma}{4\pi\varepsilon_0\sqrt{z^2+\omega^2}}$$
$$V=\int_0^R\frac{2\pi\omega\cdot d\omega\cdot\sigma}{4\pi\varepsilon_0\sqrt{z^2+\omega^2}}=\frac{\sigma}{2\varepsilon_0}(\sqrt{z^2+R^2}-z)$$

For z>>R

$$\sqrt{z^2+R^2}=z\sqrt{1+(\frac{R}{z})^2}=z(1+\frac{1}{2}\frac{R^2}{z^2}+\ldots)$$
$$V(z)=\frac{\sigma}{2\varepsilon_0}(\sqrt{R^2+z^2}-z)$$
$$=\frac{\sigma}{2\varepsilon_0}(z+\frac{R^2}{2z}-z)=\frac{\sigma}{2\varepsilon_0}\cdot\frac{R^2}{2z}=\frac{\sigma\cdot\pi R^2}{4\pi\varepsilon_0 z}$$
$$=\frac{q}{4\pi\varepsilon_0 z}$$

As point charge



Lecture 5, ACT 4

Two charged balls are each at the same potential V. Ball 2 is twice as large as ball 1.

As V is increased, which ball will induce breakdown first?

(a) Ball 1 (b) Ball 2 (c) Same Time

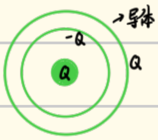
$$E_{\text{surface}}=\frac{Q}{4\pi\varepsilon_0 r^2};\ V=\frac{Q}{4\pi\varepsilon_0 r}$$

\ $E_{\text{surface}}=\frac{V}{r}$ Smaller r higher E closer to breakdown

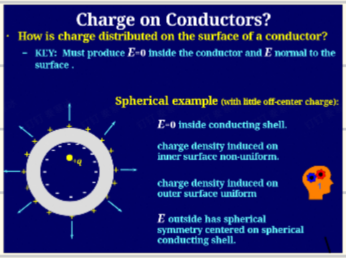
Ex. $V=100\text{ kV}$

$$r>\frac{100\bullet10^3\text{V}}{3\bullet10^6\text{V/m}}\approx0.03\text{m}=3\text{cm}$$

High Voltage Terminals must be big!



静电感应



Charge on Conductors?

- How is charge distributed on the surface of a conductor?
  - KEY: Must produce E=0 inside the conductor and E normal to the surface.

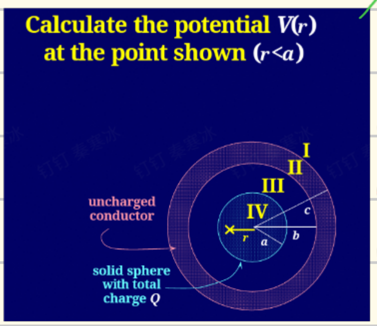
Spherical example (with little off-center charge):

E=0 inside conducting shell.

charge density induced on inner surface non-uniform.

charge density induced on outer surface uniform

E outside has spherical symmetry centered on spherical conducting shell.



Calculate the potential $V(r)$ at the point shown ($r<a$)

(1). $r>c$, $V_r=\int_r^\infty\vec{E}\cdot d\vec{l}=\frac{1}{4\pi\varepsilon_0}(\frac{1}{r}-\frac{1}{\infty})=\frac{Q}{4\pi\varepsilon_0 r}$

(2). $b<r<c$, $V_r=\int_r^\infty\vec{E}\cdot d\vec{l}=\int_r^c\vec{E}\cdot d\vec{l}+\int_c^\infty\vec{E}\cdot d\vec{l}=\int_c^\infty\vec{E}\cdot d\vec{l}=\frac{Q}{4\pi\varepsilon_0 c}$

(3). $a<r<b$, $V_r=\int_r^\infty\vec{E}\cdot d\vec{l}=\int_r^b\vec{E}\cdot d\vec{l}+\int_b^c\vec{E}\cdot d\vec{l}+\int_c^\infty\vec{E}\cdot d\vec{l}=\frac{Q}{4\pi\varepsilon_0}(\frac{1}{r}-\frac{1}{b})+\frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{c}=\frac{Q}{4\pi\varepsilon_0}[\frac{1}{r}-(\frac{1}{b}-\frac{1}{c})]$

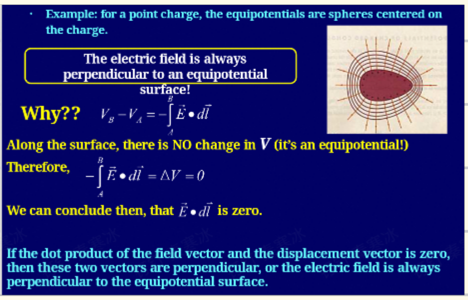
(4). $r<a$, $V_r=\int_r^\infty\vec{E}\cdot d\vec{l}=\int_r^a\vec{E}\cdot d\vec{l}+\int_a^b\vec{E}\cdot d\vec{l}+\int_b^c\vec{E}\cdot d\vec{l}+\int_c^\infty\vec{E}\cdot d\vec{l}=\frac{Q}{4\pi\varepsilon_0}\cdot(\frac{1}{a}-\frac{1}{b})+\frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{c}+\int_r^a\vec{E}\cdot d\vec{l}$

$r\to a$: $\frac{r^3}{a^3}=\frac{q}{Q}\Rightarrow q=\frac{r^3}{a^3}Q$ $\therefore\oint\vec{E}\cdot d\vec{A}=E\cdot4\pi r^2=\frac{\frac{r^3}{a^3}Q}{\varepsilon_0}\Rightarrow E=\frac{rQ}{4\pi\varepsilon_0 a^3}$

$=\frac{Q}{4\pi\varepsilon_0}(\frac{1}{a}-\frac{1}{b}+\frac{1}{c})$

$+\frac{Q}{4\pi\varepsilon_0 a}\cdot\frac{1}{2}(1-\frac{r^2}{a^2})$

- Equipotentials (等势面): The locus of points with the same potential.



- Example: for a point charge, the equipotentials are spheres centered on the charge.

The electric field is always perpendicular to an equipotential surface!

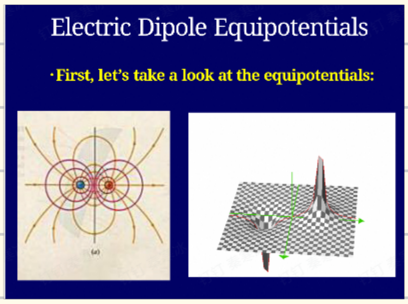
Why?? $V_B-V_A=-\int_A^B\vec{E}\bullet d\vec{l}$

Along the surface, there is NO change in V (it's an equipotential!)

Therefore, $-\int_A^B\vec{E}\bullet d\vec{l}=\Delta V=0$

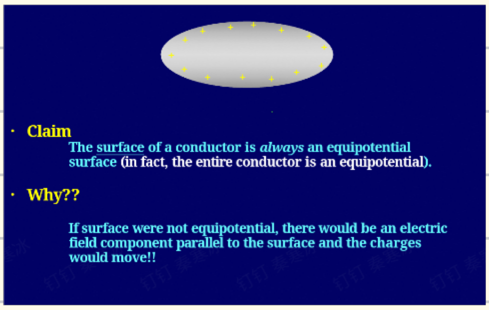
We can conclude then, that $\vec{E}\bullet d\vec{l}$ is zero.

If the dot product of the field vector and the displacement vector is zero, then these two vectors are perpendicular, or the electric field is always perpendicular to the equipotential surface.

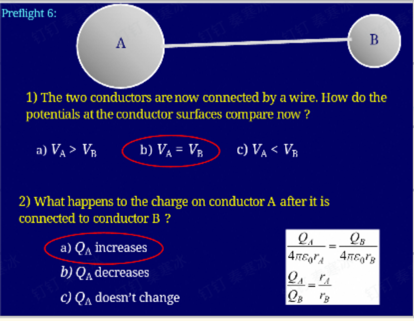


Electric Dipole Equipotentials

- First, let's take a look at the equipotentials:



- **Claim**
  The surface of a conductor is *always* an equipotential surface (in fact, the entire conductor is an equipotential).
- **Why??**
  If surface were not equipotential, there would be an electric field component parallel to the surface and the charges would move!!



Preflight 6:

1) The two conductors are now connected by a wire. How do the potentials at the conductor surfaces compare now ?

a) $V_A > V_B$ [✓ b) $V_A = V_B$] c) $V_A < V_B$

2) What happens to the charge on conductor A after it is connected to conductor B ?

[✓ a) Q_A increases]

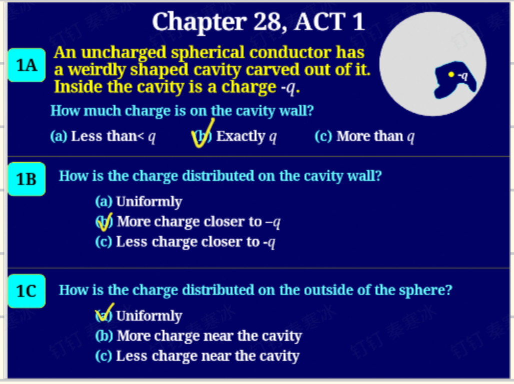
b) Q_A decreases

c) Q_A doesn't change

$$\frac{Q_A}{4\pi\varepsilon_0 r_A} = \frac{Q_B}{4\pi\varepsilon_0 r_B}$$

$$\frac{Q_A}{Q_B} = \frac{r_A}{r_B}$$

电荷均匀分布于表面，等势，$V_A=V_B$



Chapter 28, ACT 1

1A An uncharged spherical conductor has a weirdly shaped cavity carved out of it. Inside the cavity is a charge -*q*.

How much charge is on the cavity wall?

(a) Less than< *q* [✓ (b) Exactly *q*] (c) More than *q*

1B How is the charge distributed on the cavity wall?

(a) Uniformly

[✓ (b) More charge closer to –*q*]

(c) Less charge closer to -*q*

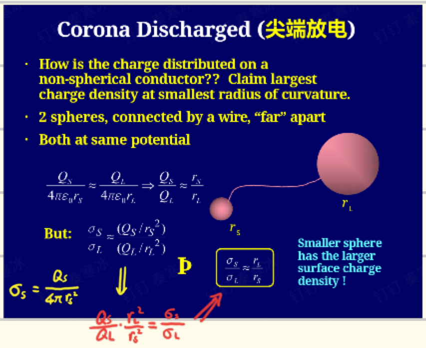
1C How is the charge distributed on the outside of the sphere?

[✓ (a) Uniformly]

(b) More charge near the cavity

(c) Less charge near the cavity

★



Corona Discharged (尖端放电)

- How is the charge distributed on a non-spherical conductor?? Claim largest charge density at smallest radius of curvature.
- 2 spheres, connected by a wire, "far" apart
- Both at same potential

$$\frac{Q_S}{4\pi\varepsilon_0 r_S} \approx \frac{Q_L}{4\pi\varepsilon_0 r_L} \Rightarrow \frac{Q_S}{Q_L} \approx \frac{r_S}{r_L}$$

But: $\frac{\sigma_S}{\sigma_L} \approx \frac{(Q_S/r_S^2)}{(Q_L/r_L^2)}$ Þ $\boxed{\frac{\sigma_S}{\sigma_L} \approx \frac{r_L}{r_S}}$

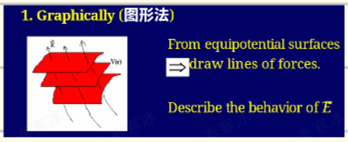
Smaller sphere has the larger surface charge density !

$\sigma_s = \frac{Q_s}{4\pi r_s^2}$ ⇓

$\frac{Q_S}{Q_L}\cdot\frac{r_L^2}{r_S^2} = \frac{\sigma_S}{\sigma_L}$ ↗

$V \Rightarrow \vec{E}$ $\vec{E} \to V$, $V_p = \int_p^\infty \vec{E}\cdot d\vec{l}$

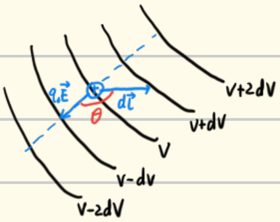
$V \to \vec{E}$?



1. Graphically (图形法)

From equipotential surfaces ⇒draw lines of forces.

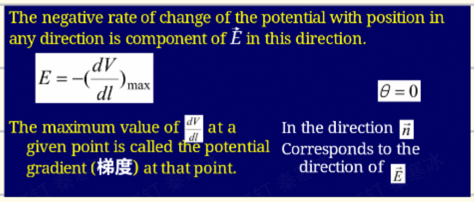
Describe the behavior of *E*



$$\boxed{dW = -q_0 dV}$$

$$\boxed{dW = \vec{F}\cdot d\vec{l} = q_0\vec{E}\cdot d\vec{l} = q_0 E\, dl\cos\theta}$$

$\therefore -q_0 dV = q_0 E\, dl\cos\theta \Rightarrow E\cos\theta = -\frac{dV}{dl} \Rightarrow \boxed{E_l = -\frac{dV}{dl}}$



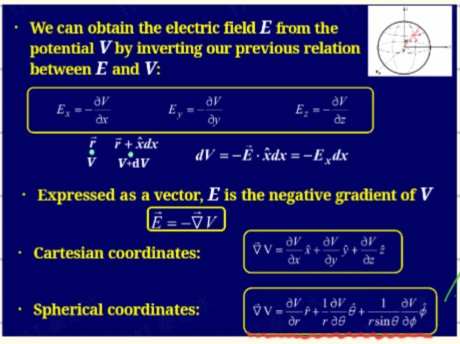
The negative rate of change of the potential with position in any direction is component of $\vec{E}$ in this direction.

$$E = -\left(\frac{dV}{dl}\right)_{\max}$$

$\theta = 0$

The maximum value of $\frac{dV}{dl}$ at a given point is called the potential gradient (梯度) at that point.

In the direction $\vec{n}$ Corresponds to the direction of $\vec{E}$



- We can obtain the electric field *E* from the potential *V* by inverting our previous relation between *E* and *V*:

$$\boxed{E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}}$$

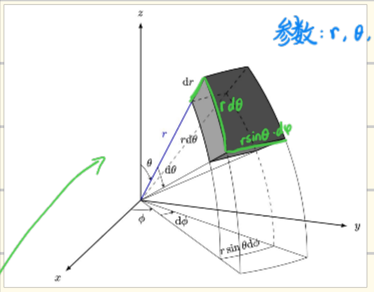
$\vec{r}$ V $\quad \vec{r}+\hat{x}dx$ V+dV $\quad dV = -\vec{E}\cdot\hat{x}dx = -E_x dx$

- Expressed as a vector, *E* is the negative gradient of *V*

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

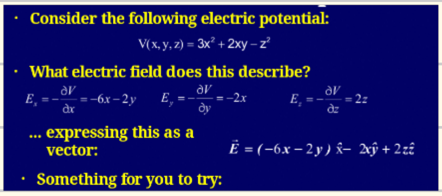
- Cartesian coordinates: $\boxed{\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}}$

- Spherical coordinates: $\boxed{\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}}$



参数：r，θ，φ

eg.



- Consider the following electric potential:

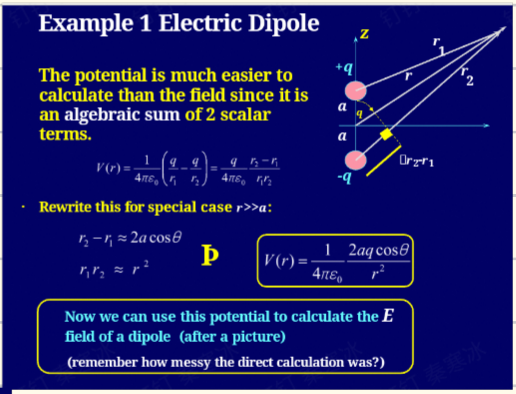
$$V(x,y,z) = 3x^2 + 2xy - z^2$$

- What electric field does this describe?

$E_x = -\frac{\partial V}{\partial x} = -6x - 2y \qquad E_y = -\frac{\partial V}{\partial y} = -2x \qquad E_z = -\frac{\partial V}{\partial z} = 2z$

... expressing this as a vector: $\vec{E} = (-6x-2y)\,\hat{x} - 2x\hat{y} + 2z\hat{z}$

- Something for you to try:



Example 1 Electric Dipole

The potential is much easier to calculate than the field since it is an algebraic sum of 2 scalar terms.

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right) = \frac{q}{4\pi\varepsilon_0}\frac{r_2 - r_1}{r_1 r_2}$$

- Rewrite this for special case *r*>>*a*:

$r_2 - r_1 \approx 2a\cos\theta$

$r_1 r_2 \approx r^2$ Þ $\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\frac{2aq\cos\theta}{r^2}}$

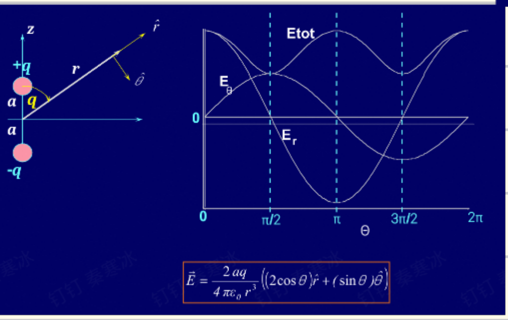
Now we can use this potential to calculate the *E* field of a dipole (after a picture)

(remember how messy the direct calculation was?)

$\Rightarrow V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2aq\cos\theta}{r^2}$

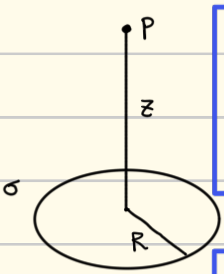
$\Rightarrow E_r = -\frac{\partial V}{\partial r} = -\frac{2aq}{4\pi\varepsilon_0}\cdot\left(\frac{-2\cos\theta}{r^3}\right)$

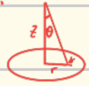
$E_\theta = -\frac{1}{r}\cdot\frac{\partial V}{\partial\theta} = -\frac{2aq}{4\pi\varepsilon_0}\left(\frac{-\sin\theta}{r^3}\right)$



$$\boxed{\vec{E} = \frac{2aq}{4\pi\varepsilon_0 r^3}\left((2\cos\theta)\hat{r} + (\sin\theta)\hat{\theta}\right)}$$

eg.　求P处的场强.



①


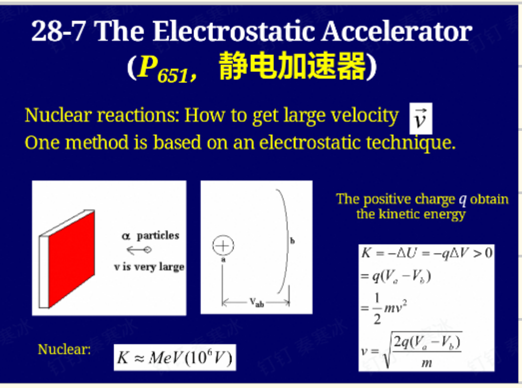
一环: $dq=\sigma\cdot 2\pi r dr$　　$\cos\theta=\frac{z}{\sqrt{z^2+r^2}}$

$\therefore dE=\frac{dq}{4\pi\varepsilon_0(z^2+r^2)}\cdot\cos\theta=\frac{2\pi z\sigma r dr}{4\pi\varepsilon_0(z^2+r^2)^{3/2}}$

$\Rightarrow E=\int dE=\int_0^R\frac{\pi z\sigma d(z^2+r^2)}{4\pi\varepsilon_0(z^2+r^2)^{3/2}}=\int_0^R\frac{z\sigma}{4\varepsilon_0(z^2+r^2)^{3/2}}d(z^2+r^2)=\frac{z\sigma}{4\varepsilon_0}\cdot\left[-2\cdot(z^2+r^2)^{-\frac{1}{2}}\right]\Big|_0^R=\frac{z\sigma}{2\varepsilon_0}\left[\frac{1}{z}-\frac{1}{\sqrt{z^2+R^2}}\right]=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{1+(R/z)^2}}\right)$

② $dq=\sigma\cdot 2\pi r dr$　$\therefore dV=\frac{\sigma\cdot 2\pi r dr}{4\pi\varepsilon_0}\cdot\frac{1}{\sqrt{z^2+r^2}}=\frac{\sigma r dr}{2\varepsilon_0\sqrt{z^2+r^2}}$　$V=\int_0^R dV=\frac{\sigma}{2\varepsilon_0}(\sqrt{z^2+R^2}-z)$

$\therefore E_z=-\frac{\partial V}{\partial z}=-\frac{\sigma}{2\varepsilon_0}\left(\frac{2z}{2\sqrt{R^2+z^2}}-1\right)=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{1+(R/z)^2}}\right)$



28-7 The Electrostatic Accelerator (*P_{651}*，静电加速器)

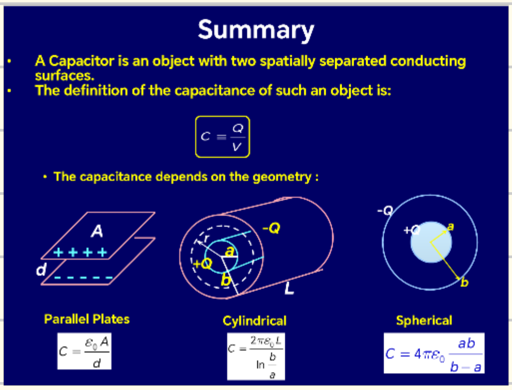
Nuclear reactions: How to get large velocity $\vec{v}$
One method is based on an electrostatic technique.

The positive charge q obtain the kinetic energy

$K=-\Delta U=-q\Delta V>0$
$=q(V_a-V_b)$
$=\frac{1}{2}mv^2$
$v=\sqrt{\frac{2q(V_a-V_b)}{m}}$

Nuclear: $K\approx MeV(10^6V)$

$V_B-V_A=\frac{W_{AB}}{q_0}$　　$V_B-V_A=-\int_A^B\vec{E}\cdot d\vec{l}$　　$V_A-V_B=\int_A^B\vec{E}\cdot d\vec{l}$



Summary

- A Capacitor is an object with two spatially separated conducting surfaces.
- The definition of the capacitance of such an object is:

$\boxed{C=\frac{Q}{V}}$

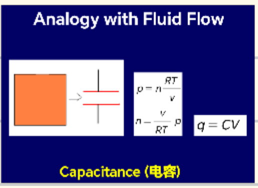
- The capacitance depends on the geometry :

Parallel Plates $C=\frac{\varepsilon_0 A}{d}$

Cylindrical $C=\frac{2\pi\varepsilon_0 L}{\ln\frac{b}{a}}$

Spherical $C=4\pi\varepsilon_0\frac{ab}{b-a}$

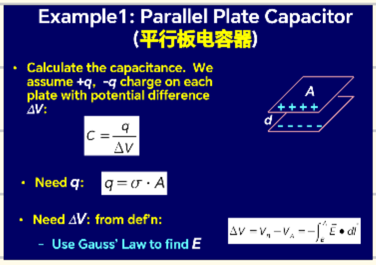
电容器　$C=\frac{q}{\Delta V}$



Analogy with Fluid Flow

$p=n\frac{RT}{V}$
$n=\frac{V}{RT}p$
$q=CV$

Capacitance (电容)



Example1: Parallel Plate Capacitor (平行板电容器)

- Calculate the capacitance. We assume +*q*, -*q* charge on each plate with potential difference *ΔV*:

$C=\frac{q}{\Delta V}$

- Need *q*: $q=\sigma\cdot A$
- Need *ΔV*: from def'n: $\Delta V=V_B-V_A=-\int_B^A\vec{E}\bullet d\vec{l}$
  - Use Gauss' Law to find *E*

而平行板电容器电场 $E=\frac{\sigma}{\varepsilon_0}$

$\therefore E=\frac{\sigma}{\varepsilon_0}=\frac{q}{A\varepsilon_0}$　　$\Delta V=V_A-V_B=\frac{q}{A\varepsilon_0}d$

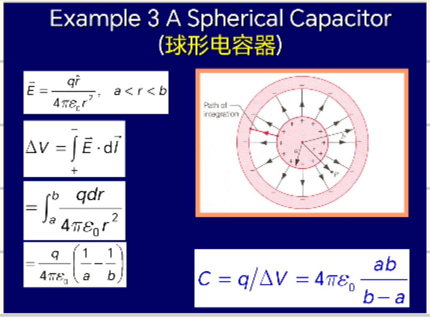
$C=\frac{q}{\Delta V}=\frac{q}{V_B-V_A}=\frac{\varepsilon_0 A}{d}$

Example 2: Cylindrical Capacitor (圆柱形电容器)

- Calculate the capacitance:
- Assume +*Q*, -*Q* on surface of cylinders with potential difference *V*.

$a<r<b$ 时的E：$E\cdot 2\pi rL=\frac{Q}{\varepsilon_0}$　$\Rightarrow E=\frac{Q}{2\pi\varepsilon_0 Lr}$

$\therefore\Delta V=-\int_b^a\vec{E}\cdot d\vec{l}=\int_a^b\vec{E}\cdot d\vec{l}=\int_a^b\frac{Q}{2\pi\varepsilon_0 Lr}dr=\frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$　　$\therefore C=\frac{Q}{\Delta V}=\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$



Example 3 A Spherical Capacitor (球形电容器)

$\vec{E}=\frac{q\hat{r}}{4\pi\varepsilon_0 r^2},\quad a<r<b$

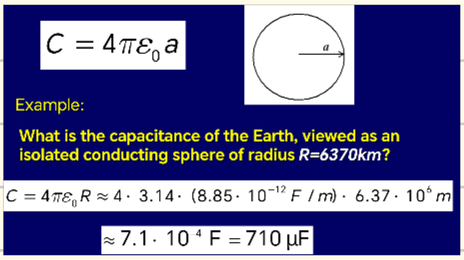
$\Delta V=\int_+^-\vec{E}\cdot d\vec{l}$

$=\int_a^b\frac{qdr}{4\pi\varepsilon_0 r^2}$

$=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$

$C=q/\Delta V=4\pi\varepsilon_0\frac{ab}{b-a}$

当 $b\to\infty$ 时, $\Delta V=\frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{a}$　$\therefore C=\frac{q}{\Delta V}=4\pi\varepsilon_0 a$　⇒ 一个导体也是电容器



$C=4\pi\varepsilon_0 a$

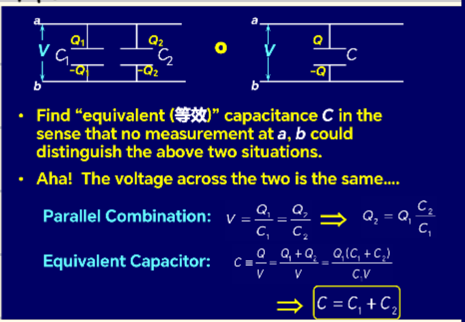
Example:

What is the capacitance of the Earth, viewed as an isolated conducting sphere of radius *R=6370km*?

$C=4\pi\varepsilon_0 R\approx 4\cdot 3.14\cdot(8.85\cdot 10^{-12}F/m)\cdot 6.37\cdot 10^6 m$

$\approx 7.1\cdot 10^{-4}\,F=710\,\mu F$

并联:

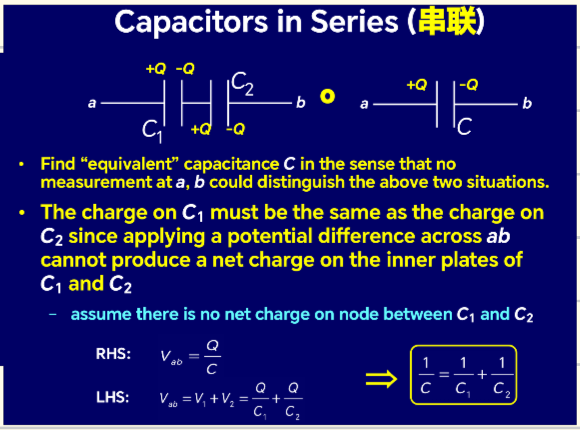


- Find "equivalent (等效)" capacitance *C* in the sense that no measurement at *a*, *b* could distinguish the above two situations.
- Aha! The voltage across the two is the same....

Parallel Combination: $V=\frac{Q_1}{C_1}=\frac{Q_2}{C_2}\Longrightarrow Q_2=Q_1\frac{C_2}{C_1}$

Equivalent Capacitor: $C\equiv\frac{Q}{V}=\frac{Q_1+Q_2}{V}=\frac{Q_1(C_1+C_2)}{C_1V}$

$\Rightarrow\boxed{C=C_1+C_2}$



Capacitors in Series (串联)

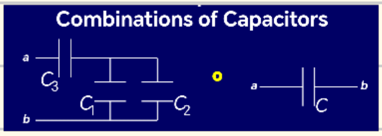
- Find "equivalent" capacitance *C* in the sense that no measurement at *a*, *b* could distinguish the above two situations.
- The charge on C_1 must be the same as the charge on C_2 since applying a potential difference across *ab* cannot produce a net charge on the inner plates of C_1 and C_2
  - assume there is no net charge on node between C_1 and C_2

RHS: $V_{ab}=\frac{Q}{C}$

LHS: $V_{ab}=V_1+V_2=\frac{Q}{C_1}+\frac{Q}{C_2}$

$\Rightarrow\boxed{\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2}}$

eg.

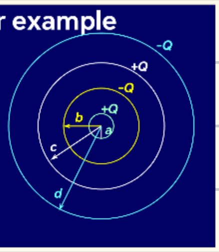


Combinations of Capacitors

$C_{12}=C_1+C_2$.

$C=\frac{C_3C_{12}}{C_3+C_{12}}=\frac{C_3(C_1+C_2)}{C_1+C_2+C_3}$

Appendix: Another example



- Suppose we have 4 concentric cylinders (圆柱) of radii *a*, *b*, *c*, *d* and charges +*Q*, -*Q*, +*Q*, -*Q*
- Question: What is the capacitance between *a* and *d*?
- Note: *E*-field between *b* and *c* is zero! WHY??
  - A cylinder of radius r_1: $b < r_1 < c$ encloses zero charge!

$$V_{ad}=\int_a^b \frac{Q\,dr}{2\pi\varepsilon_0 rL}+0+\int_c^d \frac{Q}{2\pi\varepsilon_0 rL}dr=\frac{Q}{2\pi\varepsilon_0 L}\left[\ln\left(\frac{b}{a}\right)+\ln\left(\frac{d}{c}\right)\right]$$

$$\therefore C=\frac{Q}{V_{ad}}=\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)+\ln\left(\frac{d}{c}\right)}$$

30-2 Energy storage in electric field

- How much energy is stored in a charged capacitor?
  - Calculate the work provided (usually by a battery) to charge a capacitor to +/− *Q*:

Calculate incremental work *dW* needed to add charge *dq* to capacitor at voltage *V* (there is a trick here!):

$$dW=V(q)\cdot dq=\left(\frac{q}{C}\right)\cdot dq$$

- The total work *W* to charge to *Q* is then given by:

$$\boxed{W=\frac{1}{C}\int_0^Q q\,dq=\frac{1}{2}\frac{Q^2}{C}}$$

Look at this!
Two ways to write *W*

- In terms of the voltage *V*: $\boxed{W=\frac{1}{2}CV^2}$

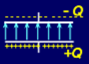
总功等于储存在电容器里面的能量

$$W=U=\frac{1}{2}CV^2$$

$$\text{能量密度}\ u=\frac{U}{V_{体}}=\frac{\frac{1}{2}CV^2}{A\cdot d}=\frac{\frac{1}{2}\cdot\frac{A\varepsilon_0}{d}V^2}{Ad}=\frac{1}{2}\cdot\frac{\varepsilon_0V^2}{d^2}=\frac{1}{2}\varepsilon_0E^2$$

Where is the Energy Stored?

- Claim: energy is stored in the electric field itself. Think of the energy needed to charge the capacitor as being the energy needed to create the field.
- To calculate the energy density in the field, first consider the constant field generated by a parallel plate capacitor, where



$$U=\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}\frac{Q^2}{(A\varepsilon_0/d)}$$

- The electric field is given by:

$$E=\frac{\sigma}{\varepsilon_0}=\frac{Q}{\varepsilon_0 A}\ \Longrightarrow\ U=\frac{1}{2}E^2\varepsilon_0 Ad$$

- The energy density *u* in the field is given by:

$$\boxed{u=\frac{W}{volume}=\frac{W}{Ad}=\frac{1}{2}\varepsilon_0E^2}$$

This is the energy density, *u*, of the electric field....

Units: $\frac{J}{m^3}$

Energy Density

Claim: the expression for the energy density of the electrostatic field $\boxed{u=\frac{1}{2}\varepsilon_0E^2}$ 普适！

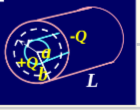
is general and is not restricted to the special case of the constant field in a parallel plate capacitor.

- **Example** (and another exercise for the student!)
  - Consider *E*-field between surfaces of cylindrical capacitor:
  - Calculate the energy in the field of the capacitor by integrating the above energy density over the volume of the space between cylinders.

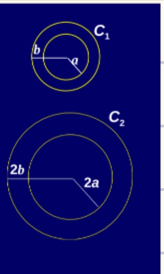
$$U=\frac{1}{2}\varepsilon_0\int E^2dv=\frac{1}{2}\varepsilon_0\int_a^b\left(\frac{\lambda}{2\pi\varepsilon_0r}\right)^2L2\pi rdr=\frac{1}{2}\frac{Q^2}{2\pi\varepsilon_0L}\ln\left(\frac{b}{a}\right)=\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}CV^2$$ ★

  - Compare this value with what you expect from the general expression:

$$\boxed{U=\frac{1}{2}CV^2}$$ 普适



Chapter 30, ACT 2



- Consider two cylindrical capacitors, each of length *L*.
  - C_1 has inner radius *a* and outer radius *b*.
  - C_2 has inner radius 2*a* and outer radius 2*b*.

If both capacitors are given the same amount of charge, what is the relation between U_1, the energy stored in C_1, and U_2, the energy stored in C_2?

(Hint: what is the relationship between C_1 and C_2?)

(a) $U_2 < U_1$ $\boxed{\text{(b) } U_2 = U_1}$ (c) $U_2 > U_1$

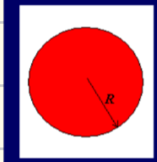
$$U=\frac{1}{2}CV^2$$

$$C=\frac{2\pi\varepsilon_0L}{\ln\left(\frac{b}{a}\right)}\Rightarrow C_1=C_2$$

$$V=\frac{Q}{C}\Rightarrow V_1=V_2$$

$$\therefore U_1=U_2$$

Problem 30-7 (page 687). An isolated conducting sphere whose radius *R* is *6.85cm* carries a charge *q=1.25nC*. (a) How much energy is stored in the electric field of this charged conductor? (b) What is the energy density (能量密度) at the surface of the sphere? (c) What is the radius R_0 of the imaginary spherical surface such that one-half of the stored potential energy lies within it?



R=6.85cm, q=1.25nC

(a) *U=?*

(b) *u=? (at the surface of the sphere)*

(c) R_0=? At $R < R_0$, *U'=1/2U*

Solution: (a)

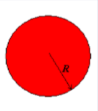
$$C=4\pi\varepsilon_0R$$

(b)

$$U=\frac{q^2}{2C}=\frac{q^2}{8\pi\varepsilon_0R}=1.03\times10^{-7}J=103nJ$$

$$E=\frac{q}{4\pi\varepsilon_0R^2}$$

$$u=\frac{1}{2}\varepsilon_0E^2=\frac{1}{2}\varepsilon_0\frac{q^2}{16\pi^2\varepsilon_0{}^2R^4}=\frac{q^2}{32\pi^2\varepsilon_0{}^2R^4}=25.4nJ/cm^3$$



$$(c)R=R_0 \qquad U(r\le R_0)=U(r\ge R_0)$$

$$\int_R^{R_0}\frac{1}{2}\varepsilon_0E^2dv=\int_{R_0}^{\infty}\frac{1}{2}\varepsilon_0E^2dv$$

$$\int_R^{R_0}\frac{1}{2}\varepsilon_0\frac{q^2}{16\pi^2\varepsilon_0{}^2r^4}4\pi r^2dr=\int_{R_0}^{\infty}\frac{1}{2}\varepsilon_0\frac{q^2}{16\pi^2\varepsilon_0{}^2r^4}4\pi r^2dr$$

$$\int_R^{R_0}\frac{dr}{r^2}=\int_{R_0}^{\infty}\frac{dr}{r^2}$$

$$\frac{1}{R}-\frac{1}{R_0}=\frac{1}{R_0}$$

$$R_0=2R=13.7cm$$

（电介质，电场中的绝缘体）

1. Capacitor with dielectrics

- **Empirical observation:**

  Inserting a non-conducting material (绝缘体) between the plates of a capacitor changes the VALUE of the capacitance.

- **Definition:**

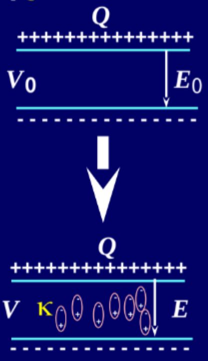
  The dielectric constant (介电常数) of a material is the ratio of the capacitance when filled with the dielectric to that without it:

$$\boxed{C=\kappa_eC_0}$$

对应中文教材的 ε_r ⇒ 相对介电常数

  - κ_e values are always > 1 (e.g., glass = 5.6; water = 78)
  - They INCREASE the capacitance of a capacitor (generally good, since it is hard to make “big” capacitors)
  - They permit more energy to be stored on a given capacitor than otherwise with vacuum (i.e., air)

Parallel Plate Example I (Q is constant)



- Deposit a charge Q on parallel plates filled with vacuum (air)—capacitance C_0.
- The potential difference is $V_0 = Q / C_0$.
- Now insert material with dielectric constant κ_e (介电常数).
  - Charge Q remains constant
  - Capacitance increases $C = \kappa_e C_0$
  - Voltage decreases from V_0 to:

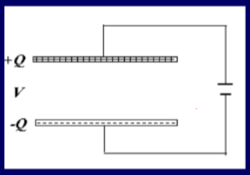
$$V=\frac{Q}{C}=\frac{Q}{\kappa_e C_0}=\frac{V_0}{\kappa_e}$$

  - Electric field decreases also:

$$E=\frac{V}{d}=\frac{V_0}{d\kappa_e}=\frac{E_0}{\kappa_e}$$

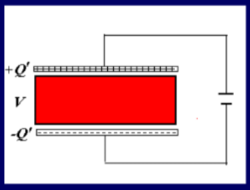
Parallel Plate Example II (V is constant)

- A charge Q on parallel plates filled with vacuum (air) and with the battery connected —capacitance C_0.
- The charge $Q = C_0V$.



- Now insert material with dielectric constant κ_e .
  - The voltage remains constant
  - Capacitance increases $C = \kappa_e C_0$
  - Charge increases from Q to:

$$Q'=\kappa_e C_0 V$$



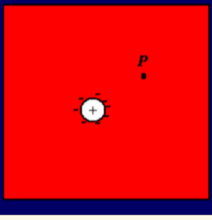
After introducing κ_e

- For a parallel-plate capacitor:
- For a cylindrical capacitor:
- For a spherical capacitor:

$$C=\frac{\kappa_e\varepsilon_0 A}{d}$$

$$C=\kappa_e\frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

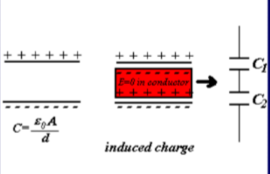
$$C=4\pi\varepsilon_0\kappa_e\frac{ab}{b-a}$$



$$E=\frac{Q}{4\pi\varepsilon_0\kappa_e r^2}$$

How to understand the increase of *C*: (Macroscopic 宏观理解)

- **The presence of conductor in a capacitor**



The redistribution of charge in a *C*

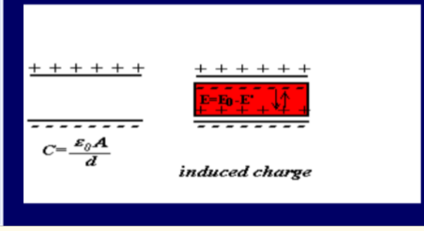
$$C_1=\frac{\varepsilon_0 A}{d_1}$$

$$C_2=\frac{\varepsilon_0 A}{d_2}$$

$$C=\frac{C_1C_2}{C_1+C_2}=\frac{\varepsilon_0 A}{d_1+d_2}>\frac{\varepsilon_0 A}{d}$$

$$(d_1+d_2<d)$$

- **The presence of a dielectrics**



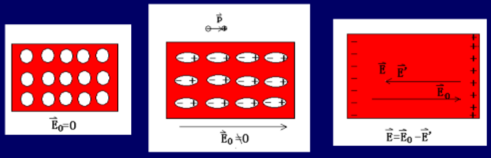
Polarization (极化)

$$V=Ed=(E_0-E')d<E_0d$$

$$C=\frac{q}{Ed}=\frac{q}{(E_0-E')d}>C_0$$

极化的微观机理：{ 无极分子电介质：$p=qd=0$ H_2、N_2、CCl_4
有极分子电介质：$p=qd\neq 0$ H_2O

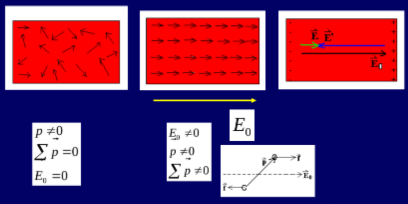
The non-polar dielectrics（无极分子电介质）: in an electric field



Induced electric dipole moment（感生电偶极矩）

Electric displacement polarization（电子位移极化）

Polar dielectrics（有极分子电介质） in an electric field

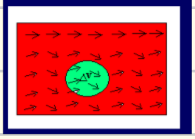


Alignment polarization（取向极化）

- Notes: In high frequency field, Electric displacement polarization（电子位移极化）plays an important role.

· Polarization (极化强度矢量P)

- **Definition**

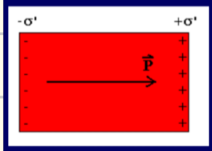


In the volume of ΔV

$\sum\vec{p}\neq0$ $\vec{P}=\frac{\sum\vec{p}_m}{\Delta V}$ ps. $\vec{P}$一定沿$\vec{E}$方向；$\vec{p}$不一定！

• Relationship between $\vec{P}$ and σ'

For uniform dielectrics



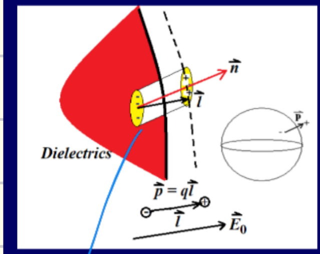
The induced charge distributes only on the surface of dielectrics.

For displacement polarization (位移极化)

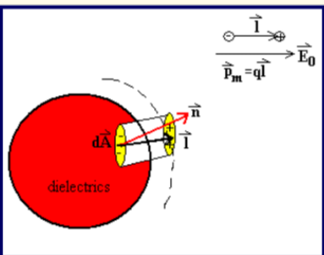
Consider the positive induced charge through area $d\vec{A}$ due to polarization.

$$\vec{P} = \frac{\sum \vec{p}_m}{\Delta V} = nq\vec{l}$$

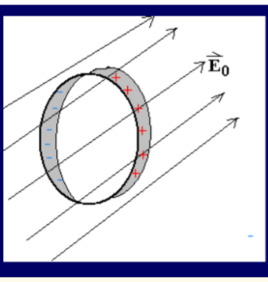
n, the number of molecular per unit volume

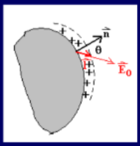


$$dN = ndV = nldA\cos\theta$$
$$dq' = qdN = nqldA\cos\theta$$
$$= PdA\cos\theta$$
$$= \vec{P}\bullet d\vec{A}$$
$$\oiint \vec{P}\bullet d\vec{A} = \sum_{out} q' = -\sum_{in} q'$$

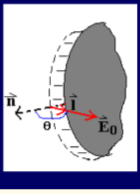


$$dq' = P\bullet dA = P\cos\theta \cdot dA$$
$$\sigma' = \frac{dq'}{dA} = P\cos\theta = \vec{P}\bullet\vec{n} = P_n$$





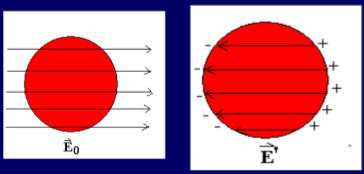
$$\theta < \frac{\pi}{2}, \sigma'_e > 0, +$$



$$\theta > \frac{\pi}{2}, \sigma'_e < 0, -$$

取l长度，内部全为正电荷

4. Depolarization Field（退极化场）



$$E = E_0 + E'$$

Depolarization field *E'*

At some place, E', E_0 in the same direction.

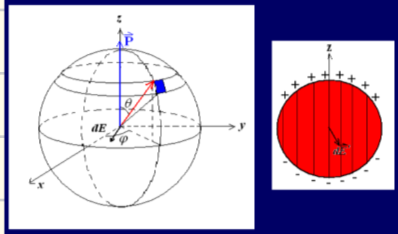
At another place, E', E_0 in the opposite direction

Example 1

A spherical dielectrics, uniform polarization $\vec{P}$

$\sigma_e' = P\cos\theta$

Calculating the depolarization field *E'* (退极化场) at center.

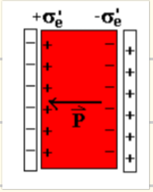


$$\sigma_e' = P_n = P\cos\theta$$
$$dE' = \frac{dq'}{4\pi\varepsilon_0 R^2} = \frac{\sigma_e' dA}{4\pi\varepsilon_0 R^2} = \frac{P\cos\theta dA}{4\pi\varepsilon_0 R^2}$$
$$dA = Rd\theta \cdot R\sin\theta d\varphi$$
$$= R^2 \sin\theta d\theta d\varphi$$

$$dE' = \frac{P}{4\pi\varepsilon_0}\cos\theta\sin\theta d\theta d\varphi$$

$$dE'_z = dE'\cos(\pi - \theta) = -dE'\cos\theta$$
$$= -\frac{P}{4\pi\varepsilon_0}\cos^2\theta\sin\theta d\theta d\varphi$$
$$E'_z = \oint\oint E'_z = -\frac{P}{4\pi\varepsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta d\theta\int_0^{2\pi} d\varphi = -\frac{P}{3\varepsilon_0}$$

eg. Parallel plate



$\sigma_e' = P\cos\theta = P$

$E' = \frac{\sigma_e'}{\varepsilon_0}$

5. Polarization law of dielectrics (电介质的极化规律)

$$P \Rightarrow \sigma_e' \Rightarrow E' \Rightarrow E$$

$P(E)$ function

For different materials, $P(E)$ different and complicated, which is determined by an experiment result.

- **For general isotropic materials（各向同性）**

$$P = \chi_e \varepsilon_0 E$$

χ_e**: Polarization coefficient （极化率）**

$\kappa_e = 1 + \chi_e$

- **For some crystal materials (anisotropic,** 如 SiO$_2$ 晶体水晶等)

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} \chi_{xx}\chi_{xy}\chi_{xz} \\ \chi_{yx}\chi_{yy}\chi_{yz} \\ \chi_{zx}\chi_{zy}\chi_{zz} \end{pmatrix} \begin{pmatrix} \varepsilon_0 E_x \\ \varepsilon_0 E_y \\ \varepsilon_0 E_z \end{pmatrix}$$

张量

$$P_x = \chi_{xx}\varepsilon_0 E_x + \chi_{xy}\varepsilon_0 E_y + \chi_{xz}\varepsilon_0 E_z$$
$$P_y = \chi_{yx}\varepsilon_0 E_x + \chi_{yy}\varepsilon_0 E_y + \chi_{yz}\varepsilon_0 E_z$$
$$P_z = \chi_{zx}\varepsilon_0 E_x + \chi_{zy}\varepsilon_0 E_y + \chi_{zz}\varepsilon_0 E_z$$

$$\left\{ \begin{array}{l} \sigma_e' = \vec{P}\cdot\vec{n} = P\cos\theta \\ \vec{P} = \frac{\sum \vec{P}_m}{\Delta V} = nq\vec{l} \end{array} \right.$$

6. Electric Displacement Vector $\vec{D}$ (电位移矢量)
and *Gauss' Law* with Dielectrics.

In dielectrics: $E \neq 0$

$$E_0 \rightarrow P \rightarrow \sigma'_e \rightarrow E' \rightarrow E = E_0 + E'$$

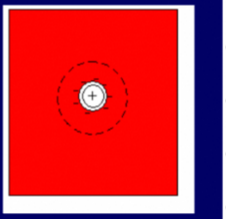
very complicated

Induce a new physical quantity:

D

Electric Displacement Vector（电位移矢量）
Or Electric Induction Vector（电感应矢量）

Electric Displacement Vector $\vec{D}$



$$\varepsilon_0 \oiint \vec{E}\bullet d\vec{A} = \sum_{In}(q_0+q')$$

$$\oiint \vec{P}\bullet d\vec{A} = -\sum_{In} q'$$

$$\oiint \varepsilon_0\vec{E}\bullet d\vec{A} = \sum_{In} q_0 - \oiint \vec{P}\bullet d\vec{A}$$

$$\oiint (\varepsilon_0\vec{E}+\vec{P})\bullet d\vec{A} = \sum_{In} q_0$$

$$\oiint \vec{D}\bullet d\vec{A} = \sum_{In} q_0$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\oiint \vec{D}\bullet d\vec{A} = \sum_{In} q_0$$

Gauss Law in the dielectrics

此处q'为极化的
电荷，因此q_0为
自由电荷！

Electric Displacement Vector (电位移矢量)

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$
$$= \varepsilon_0\vec{E} + \chi_e\varepsilon_0\vec{E}$$
$$= (1+\chi_e)\varepsilon_0\vec{E}$$
$$= \kappa_e\varepsilon_0\vec{E}$$

κ_e **Dielectric constant(介电常数)**

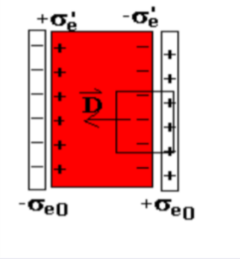
χ_e **Polarization Coefficient (极化率)**

$$\kappa_e = 1 + \chi_e$$

$$\vec{P} = \chi_e\varepsilon_0\vec{E}$$

$$\oiint \vec{D}\bullet d\vec{A} = \sum_{ins} q_0$$

- **Example 1**



$$\oint D\bullet dA = \sum q_0$$

$$\underset{\rightarrow}{D_1}\Delta A + D_2\Delta A = \sigma_{e0}\Delta A$$

$$E_1 = 0, D_1 = \kappa_{e1}\varepsilon_0 E_1 = 0, \because D_1 = 0$$

In conductor

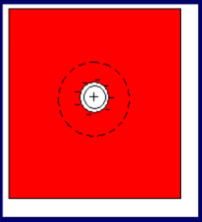
$$\therefore D = D_2 = \sigma_{e0} = \varepsilon_0 E_0$$

$$\therefore E = \frac{D}{\kappa_e\varepsilon_0} = \frac{\varepsilon_0 E_0}{\kappa_e\varepsilon_0} = \frac{E_0}{\kappa_e}$$

$$D = \varepsilon_0 E + P = \kappa_e\varepsilon_0 E$$

Example 2

$$D = \varepsilon_0 E + P = \kappa_e\varepsilon_0 E$$



$$\Box\iint D\bullet dA = \sum q_0$$

$$4\pi r^2 D = q_0$$

$$D = \frac{q_0}{4\pi r^2}$$

$$E = \frac{D}{\kappa_e\varepsilon_0} = \frac{q_0}{4\pi\varepsilon_0\kappa_e r^2} = \frac{E_0}{\kappa_e}$$

Note:

$\oint E\bullet dl = 0$, but $\oint D\bullet dl \neq 0$ Why?

· 电流强度 ： $i = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$

电流密度矢量：$\vec{j}$

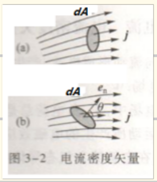


图 3-2 电流密度矢量

$di = \vec{j} \cdot d\vec{A}$

$i = \iint_A \vec{j} \cdot d\vec{A} = \iint_A j \cos\theta \, dA$

· 电流连续方程

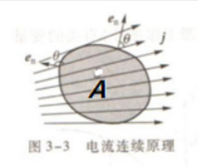


图 3-3 电流连续原理

· 电流场：$\oint\!\!\!\oint_A \vec{j} \cdot d\vec{A} = -\frac{dq}{dt}$

恒定电流条件：$\oint\!\!\!\oint_A \vec{j} \cdot d\vec{A} = 0 \Rightarrow j_1 \Delta A_1 = j_2 \Delta A_2$ 无电荷积累

(电流稳恒的条件)

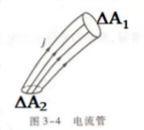
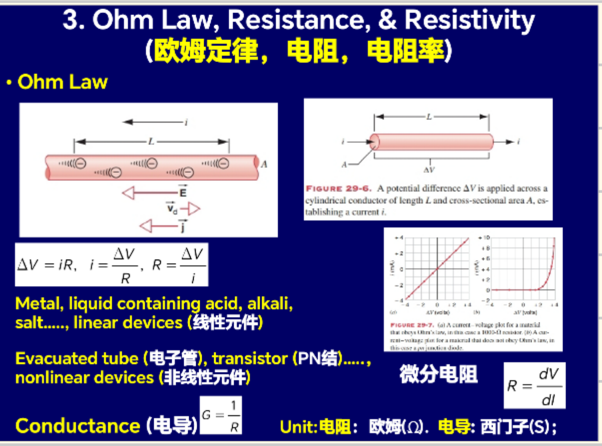


图 3-4 电流管



3. Ohm Law, Resistance, & Resistivity (欧姆定律，电阻，电阻率)

- Ohm Law

FIGURE 29-6. A potential difference ΔV is applied across a cylindrical conductor of length L and cross-sectional area A, establishing a current i.

$$\Delta V = iR, \quad i = \frac{\Delta V}{R}, \quad R = \frac{\Delta V}{i}$$

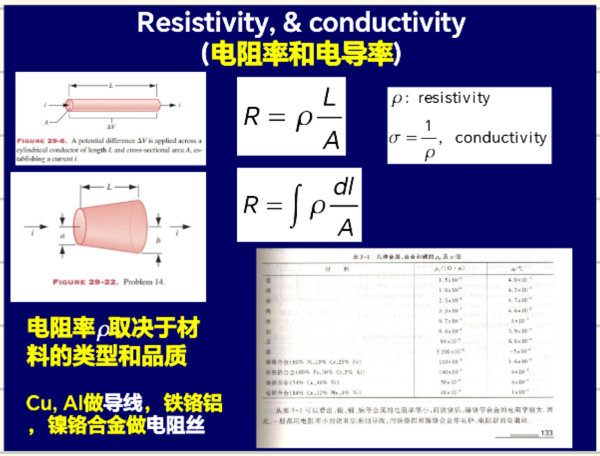
Metal, liquid containing acid, alkali, salt….., linear devices (线性元件)

FIGURE 29-7. (a) A current–voltage plot for a material that obeys Ohm's law, in this case a 1000-Ω resistor. (b) A current–voltage plot for a material that does not obey Ohm's law, in this case a pn junction diode.

Evacuated tube (电子管), transistor (PN结)….., nonlinear devices (非线性元件)

微分电阻 $R = \frac{dV}{dI}$

Conductance (电导) $G = \frac{1}{R}$ Unit:电阻：欧姆(Ω). 电导: 西门子(S);



Resistivity, & conductivity (电阻率和电导率)

FIGURE 29-6. A potential difference ΔV is applied across a cylindrical conductor of length L and cross-sectional area A, establishing a current i.

$$R = \rho \frac{L}{A}$$

ρ: resistivity

$\sigma = \frac{1}{\rho}$, conductivity

FIGURE 29-22. Problem 14.

$$R = \int \rho \frac{dl}{A}$$

电阻率ρ取决于材料的类型和品质

Cu, Al做导线，铁铬铝，镍铬合金做电阻丝

欧姆定律微分形式

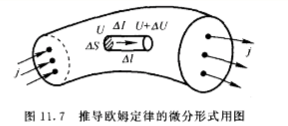
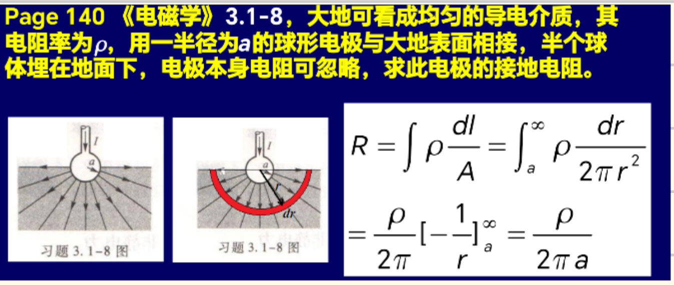


图 11.7 推导欧姆定律的微分形式用图

$E = \frac{\Delta V}{\Delta l} \quad \therefore E\Delta l = \Delta V = j \cdot \Delta S \cdot \Delta R = j \cdot \Delta S \cdot \rho \cdot \frac{\Delta l}{\Delta S}$

$= j\rho\Delta l \Rightarrow \vec{j} = \frac{1}{\rho}\vec{E}$

$\Rightarrow \boxed{\vec{j} = \gamma \vec{E}}$



Page 140 《电磁学》3.1-8，大地可看成均匀的导电介质，其电阻率为ρ，用一半径为a的球形电极与大地表面相接，半个球体埋在地面下，电极本身电阻可忽略，求此电极的接地电阻。

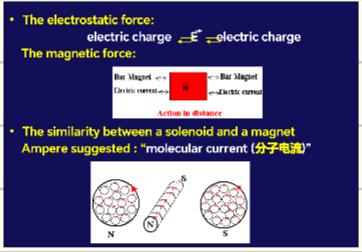
习题 3. 1-8 图

习题 3. 1-8 图

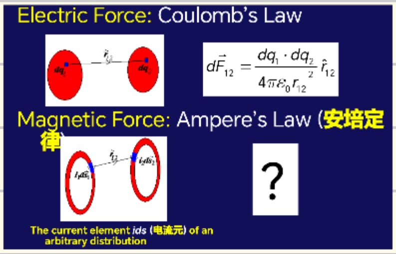
$$R = \int \rho \frac{dl}{A} = \int_a^\infty \rho \frac{dr}{2\pi r^2} = \frac{\rho}{2\pi}[-\frac{1}{r}]_a^\infty = \frac{\rho}{2\pi a}$$

焦耳定律微分形式：$\Delta P = (\Delta I)^2 \Delta R = (j\Delta S)^2 (\rho \frac{\Delta l}{\Delta S}) = \rho j^2 \Delta V = \frac{1}{\rho} E^2 \Delta V = \gamma E^2 \Delta V$ 热功率密度：$w = \frac{\Delta P}{\Delta V} = \gamma E^2$

·稳恒磁场



- The electrostatic force:
  electric charge ⇄ $\vec{E}$ ⇄ electric charge
  The magnetic force:
- The similarity between a solenoid and a magnet
  Ampere suggested : "molecular current (分子电流)"



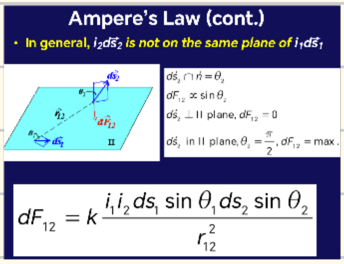
Electric Force: Coulomb's Law

$$d\vec{F}_{12} = \frac{dq_1 \cdot dq_2}{4\pi\varepsilon_0 r_{12}^2}\hat{r}_{12}$$

Magnetic Force: Ampere's Law (安培定律)

?

The current element *ids* (电流元) of an arbitrary distribution



Ampere's Law (cont.)

- In general, $i_2d\vec{s}_2$ *is not on the same plane of* $i_1d\vec{s}_1$

$d\vec{s}_2 \cap \vec{n} = \theta_2$

$dF_{12} \propto \sin\theta_2$

$d\vec{s}_2 \perp$ II plane, $dF_{12} = 0$

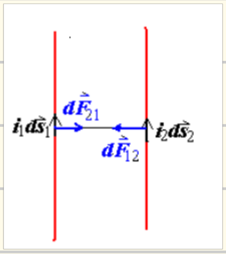
$d\vec{s}_2$ in II plane, $\theta_2 = \frac{\pi}{2}$, $dF_{12} = \max$.

$$dF_{12} = k\frac{i_1 i_2 ds_1 \sin\theta_1 ds_2 \sin\theta_2}{r_{12}^2}$$

安培定理：

$d\vec{F}_{12} = k\cdot\frac{i_2d\vec{s}_2\times(i_1d\vec{s}_1\times\hat{r}_{12})}{r_{12}^2}$ $K = \frac{\mu_0}{4\pi} = 10^{-7}N/A^2$ $\mu_0 = 4\pi\times10^{-7}N/A^2$ (磁导率)

eg. 1

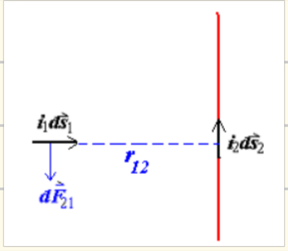


$d\vec{F}_{12} = \frac{\mu_0}{4\pi}\cdot\frac{i_2d\vec{s}_2\times(i_1d\vec{s}_1\times\hat{r}_{12})}{r_{12}^2}$ $d\vec{s}_1\perp\hat{r}_{12}$ $\therefore dF_{12} = \frac{\mu_0}{4\pi}\cdot\frac{i_1i_2ds_1ds_2}{r_{12}^2}$

$d\vec{F}_{21} = \frac{\mu_0}{4\pi}\cdot\frac{i_1d\vec{s}_1\times(i_2d\vec{s}_2\times\hat{r}_{21})}{r_{21}^2}$ $d\vec{s}_2\perp\hat{r}_{21}$ $\therefore dF_{21} = \frac{\mu_0}{4\pi}\cdot\frac{i_1i_2ds_1ds_2}{r_{12}^2}$

$dF_{12} = dF_{21}$

eg 2.

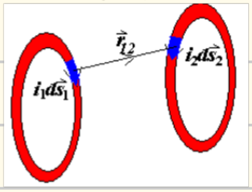


$d\vec{F}_{12} = \frac{\mu_0}{4\pi}\cdot\frac{i_2d\vec{s}_2\times(i_1d\vec{s}_1\times\hat{r}_{12})}{r_{12}^2}$ $d\vec{s}_1 // \hat{r}_{12}$ $\therefore dF_{12} = 0$

$d\vec{F}_{21} = \frac{\mu_0}{4\pi}\cdot\frac{i_1d\vec{s}_1\times(i_2d\vec{s}_2\times\hat{r}_{21})}{r_{21}^2}$ $d\vec{s}_2\perp\hat{r}_{21}$ $\therefore dF_{21} = \frac{\mu_0}{4\pi}\cdot\frac{i_1i_2ds_1ds_2}{r_{12}^2}$

$dF_{12}\neq dF_{21}$ （牛顿第三定律在宏观上成立！）

·磁感应强度B



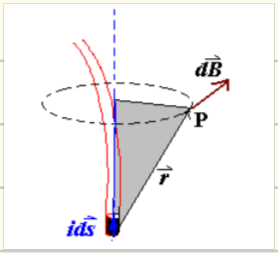
$d\vec{F}_{12} = \frac{\mu_0}{4\pi}\cdot\frac{i_2d\vec{s}_2\times(i_1d\vec{s}_1\times\hat{r}_{12})}{r_{12}^2} \Rightarrow d\vec{F}_2 = \frac{\mu_0}{4\pi}\cdot i_2d\vec{s}_2\times\oint_{L_1}\frac{i_1d\vec{s}_1\times\hat{r}_{12}}{r_{12}^2}$

$\boxed{\therefore \text{定义: } \vec{B}_1 = \frac{\mu_0}{4\pi}\oint_{L_1}\frac{i_1d\vec{s}_1\times\hat{r}_{12}}{r_{12}^2} \quad \therefore d\vec{F}_2 = i_2d\vec{s}_2\times\vec{B}_1}$

安培力

Unit : Tesla (T) ; 1T = 1N/m.A, 1T = 10^4 Gauss

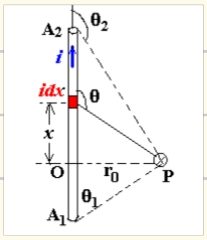
• Biot-Savart (毕奥-萨阀尔)定理.



$$d\vec{B} = \frac{\mu_0}{4\pi}\cdot\frac{id\vec{s}\times\hat{r}}{r^2}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\oint_L\frac{id\vec{s}\times\hat{r}}{r^2}$$

A long straight line



$dB = \frac{\mu_0}{4\pi}\cdot\frac{idx\sin\theta}{r^2}$

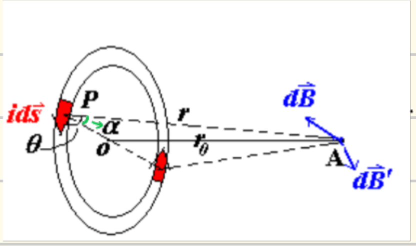
P处: $B = \int_{A_1}^{A_2}dB = \frac{\mu_0}{4\pi}\int_{A_1}^{A_2}\frac{i\sin\theta dx}{r^2}$ $r_0 = r\sin(\pi-\theta) = r\sin\theta$

$r = \frac{r_0}{\sin\theta}$ $x = -\frac{r_0}{\tan\theta} \Rightarrow dx = \frac{r_0d\theta}{\sin^2\theta}$

$B = \int_{\theta_1}^{\theta_2}\frac{\mu_0 i}{4\pi}\frac{\sin\theta\cdot\frac{r_0d\theta}{\sin^2\theta}}{r_0^2/\sin^2\theta} = \frac{\mu_0 i}{4\pi r_0}(\cos\theta_1-\cos\theta_2)$

⇒ 无穷长: $\theta_1=0$, $\cos\theta_1=1$; $\theta_2=\pi$, $\cos\theta_2=-1$ $\therefore \boxed{B = \frac{\mu_0 i}{2\pi r_0}}$

· circular loop



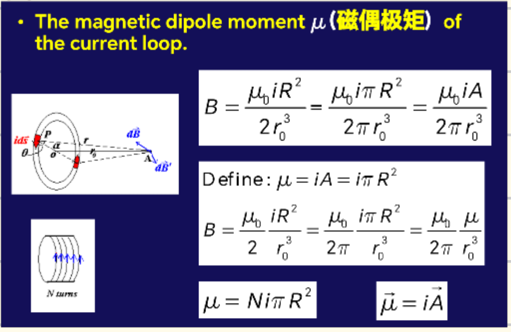
$|d\vec{B}| = |d\vec{B}'|$ $d\vec{B} = \frac{\mu_0}{4\pi}\cdot\frac{id\vec{s}\times\hat{r}}{r^2}$ $dB_x = dB\cdot\cos\alpha$ $dB = \frac{\mu_0}{4\pi}\cdot\frac{ids}{r^2}\sin\theta$ $r = r_0/\sin\alpha$

$\therefore B_x = \oint dB_x = \oint dB\cos\alpha \Rightarrow B = \frac{\mu_0 i}{4\pi}\oint\frac{\sin^2\alpha}{r_0^2}\cos\alpha\, ds = \frac{\mu_0 i}{4\pi r_0^2}\sin^2\alpha\cos\alpha\cdot 2\pi R$

$= \boxed{\frac{\mu_0}{2}\cdot\frac{iR^2}{(R^2+r_0^2)^{3/2}}}$

·At the center of the loop $(r_0=0)$ $B = \frac{\mu_0 i}{2R}$

· $r_0 >> R$; $B = \frac{\mu_0 i R^2}{2r_0^3}$



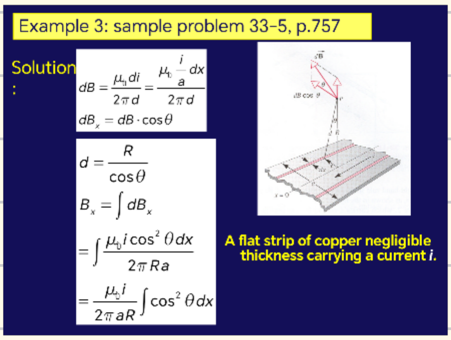
- The magnetic dipole moment μ (磁偶极矩) of the current loop.

$$B=\frac{\mu_0 iR^2}{2r_0^3}=\frac{\mu_0 i\pi R^2}{2\pi r_0^3}=\frac{\mu_0 iA}{2\pi r_0^3}$$

Define: $\mu = iA = i\pi R^2$

$$B=\frac{\mu_0}{2}\frac{iR^2}{r_0^3}=\frac{\mu_0}{2\pi}\frac{i\pi R^2}{r_0^3}=\frac{\mu_0}{2\pi}\frac{\mu}{r_0^3}$$

$\mu = Ni\pi R^2$ $\qquad$ $\vec{\mu}=i\vec{A}$



Example 3: sample problem 33-5, p.757

Solution:

$$dB=\frac{\mu_0 di}{2\pi d}=\frac{\mu_0 \frac{i}{a}dx}{2\pi d}$$

$$dB_x = dB\cdot\cos\theta$$

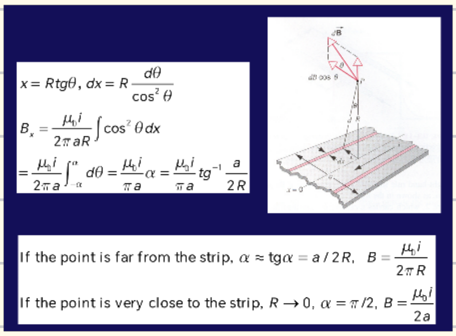
$$d=\frac{R}{\cos\theta}$$

$$B_x=\int dB_x$$

$$=\int\frac{\mu_0 i\cos^2\theta dx}{2\pi Ra}$$

$$=\frac{\mu_0 i}{2\pi aR}\int\cos^2\theta dx$$

A flat strip of copper negligible thickness carrying a current *i*.



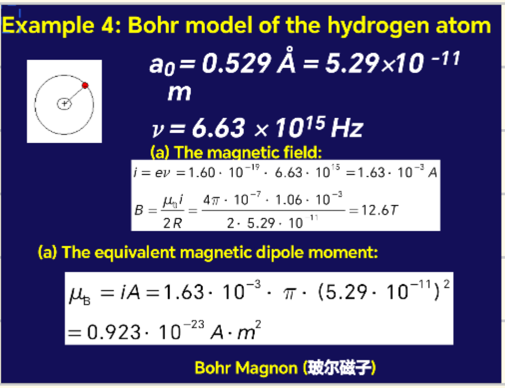
$$x=Rtg\theta,\ dx=R\frac{d\theta}{\cos^2\theta}$$

$$B_x=\frac{\mu_0 i}{2\pi aR}\int\cos^2\theta dx$$

$$=\frac{\mu_0 i}{2\pi a}\int_{-\alpha}^{\alpha}d\theta=\frac{\mu_0 i}{\pi a}\alpha=\frac{\mu_0 i}{\pi a}tg^{-1}\frac{a}{2R}$$

If the point is far from the strip, $\alpha\approx \mathrm{tg}\alpha = a/2R$, $B=\frac{\mu_0 i}{2\pi R}$

If the point is very close to the strip, $R\to 0$, $\alpha=\pi/2$, $B=\frac{\mu_0 i}{2a}$



Example 4: Bohr model of the hydrogen atom

$a_0 = 0.529$ Å $= 5.29\times10^{-11}$ m

$\nu = 6.63\times10^{15}$ Hz

(a) The magnetic field:

$$i=e\nu=1.60\cdot10^{-19}\cdot6.63\cdot10^{15}=1.63\cdot10^{-3}A$$

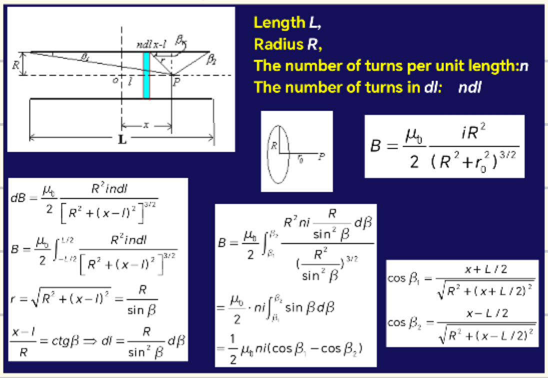
$$B=\frac{\mu_0 i}{2R}=\frac{4\pi\cdot10^{-7}\cdot1.06\cdot10^{-3}}{2\cdot5.29\cdot10^{-11}}=12.6T$$

(a) The equivalent magnetic dipole moment:

$$\mu_B=iA=1.63\cdot10^{-3}\cdot\pi\cdot(5.29\cdot10^{-11})^2$$
$$=0.923\cdot10^{-23}A\cdot m^2$$

Bohr Magnon (玻尔磁子)

Solenoid (螺线管)



Length *L*,
Radius *R*,
The number of turns per unit length: *n*
The number of turns in *dl*: *ndl*

$$B=\frac{\mu_0}{2}\frac{iR^2}{(R^2+r_0^2)^{3/2}}$$

$$dB=\frac{\mu_0}{2}\frac{R^2indl}{\left[R^2+(x-l)^2\right]^{3/2}}$$

$$B=\frac{\mu_0}{2}\int_{-L/2}^{L/2}\frac{R^2indl}{\left[R^2+(x-l)^2\right]^{3/2}}$$

$$r=\sqrt{R^2+(x-l)^2}=\frac{R}{\sin\beta}$$

$$\frac{x-l}{R}=ctg\beta\Rightarrow dl=\frac{R}{\sin^2\beta}d\beta$$

$$B=\frac{\mu_0}{2}\int_{\beta_1}^{\beta_2}\frac{R^2ni\frac{R}{\sin^2\beta}d\beta}{(\frac{R^2}{\sin^2\beta})^{3/2}}$$

$$=\frac{\mu_0}{2}\cdot ni\int_{\beta_1}^{\beta_2}\sin\beta d\beta$$

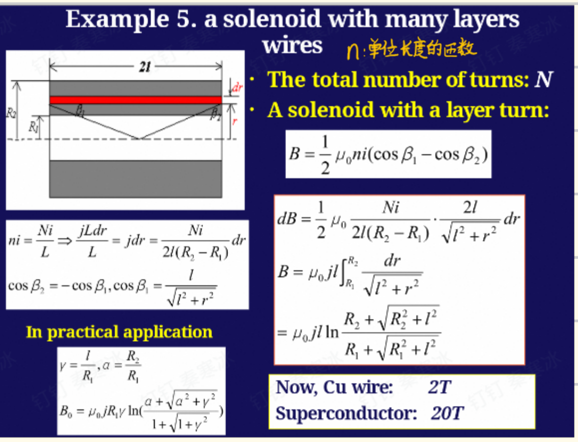
$$=\frac{1}{2}\mu_0 ni(\cos\beta_1-\cos\beta_2)$$

$$\cos\beta_1=\frac{x+L/2}{\sqrt{R^2+(x+L/2)^2}}$$

$$\cos\beta_2=\frac{x-L/2}{\sqrt{R^2+(x-L/2)^2}}$$

$L\to\infty,\ \beta_1=0,\ \beta_2=\pi$

$B=\frac{1}{2}\mu_0 ni(1+1)=\mu_0 ni$



Example 5. a solenoid with many layers wires n:单位长度的函数

- The total number of turns: *N*
- A solenoid with a layer turn:

$$B=\frac{1}{2}\mu_0 ni(\cos\beta_1-\cos\beta_2)$$

$$ni=\frac{Ni}{L}\Rightarrow\frac{jLdr}{L}=jdr=\frac{Ni}{2l(R_2-R_1)}dr$$

$$\cos\beta_2=-\cos\beta_1,\cos\beta_1=\frac{l}{\sqrt{l^2+r^2}}$$

$$dB=\frac{1}{2}\mu_0\frac{Ni}{2l(R_2-R_1)}\cdot\frac{2l}{\sqrt{l^2+r^2}}dr$$

$$B=\mu_0 jl\int_{R_1}^{R_2}\frac{dr}{\sqrt{l^2+r^2}}$$

$$=\mu_0 jl\ln\frac{R_2+\sqrt{R_2^2+l^2}}{R_1+\sqrt{R_1^2+l^2}}$$

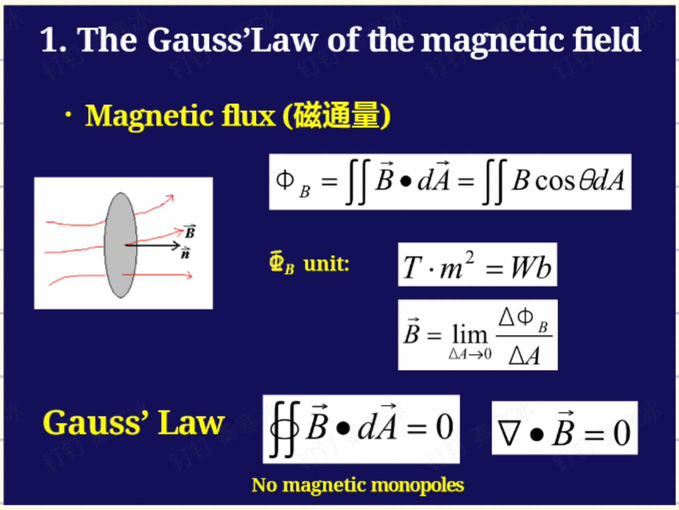
In practical application

$$\gamma=\frac{l}{R_1},\alpha=\frac{R_2}{R_1}$$

$$B_0=\mu_0 jR_1\gamma\ln(\frac{\alpha+\sqrt{\alpha^2+\gamma^2}}{1+\sqrt{1+\gamma^2}})$$

Now, Cu wire: *2T*
Superconductor: *20T*

5. 磁通量



1. The Gauss'Law of the magnetic field

- **Magnetic flux (磁通量)**

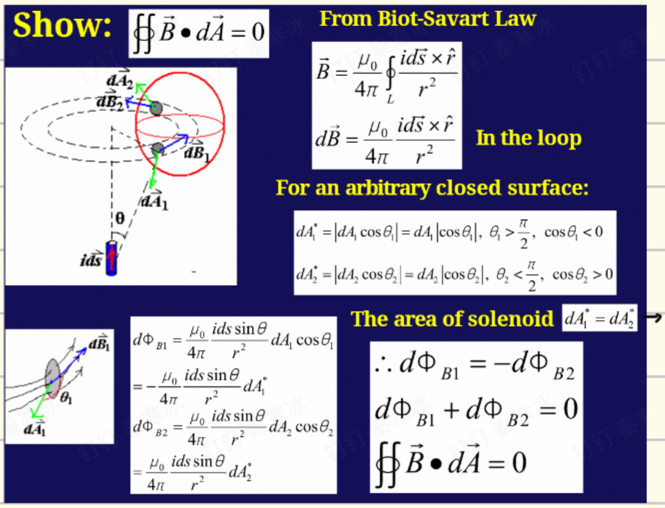
$$\Phi_B=\iint\vec{B}\bullet d\vec{A}=\iint B\cos\theta dA$$

Φ_B unit: $T\cdot m^2=Wb$

$$\vec{B}=\lim_{\Delta A\to0}\frac{\Delta\Phi_B}{\Delta A}$$

Gauss' Law $\oiint\vec{B}\bullet d\vec{A}=0$ $\qquad$ $\nabla\bullet\vec{B}=0$

No magnetic monopoles



Show: $\oiint\vec{B}\bullet d\vec{A}=0$

From Biot-Savart Law

$$\vec{B}=\frac{\mu_0}{4\pi}\oint_L\frac{id\vec{s}\times\hat{r}}{r^2}$$

$$d\vec{B}=\frac{\mu_0}{4\pi}\frac{id\vec{s}\times\hat{r}}{r^2}$$ In the loop

For an arbitrary closed surface:

$$dA_1^*=|dA_1\cos\theta_1|=dA_1|\cos\theta_1|,\ \theta_1>\frac{\pi}{2},\ \cos\theta_1<0$$

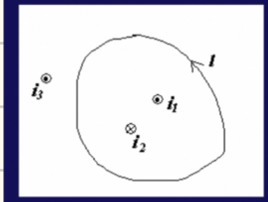
$$dA_2^*=|dA_2\cos\theta_2|=dA_2|\cos\theta_2|,\ \theta_2<\frac{\pi}{2},\ \cos\theta_2>0$$

$$d\Phi_{B1}=\frac{\mu_0}{4\pi}\frac{ids\sin\theta}{r^2}dA_1\cos\theta_1$$
$$=-\frac{\mu_0}{4\pi}\frac{ids\sin\theta}{r^2}dA_1^*$$
$$d\Phi_{B2}=\frac{\mu_0}{4\pi}\frac{ids\sin\theta}{r^2}dA_2\cos\theta_2$$
$$=\frac{\mu_0}{4\pi}\frac{ids\sin\theta}{r^2}dA_2^*$$

The area of solenoid $dA_1^*=dA_2^*$ →大小相等

$$\therefore d\Phi_{B1}=-d\Phi_{B2}$$
$$d\Phi_{B1}+d\Phi_{B2}=0$$
$$\oiint\vec{B}\bullet d\vec{A}=0$$

2. The Ampere's Loop Law of a magnetic field
(磁场安培环路定律)



$$\oint \vec{B} \bullet d\vec{l} = \mu_0 \sum_{in\ loop} i$$

$\sum i$ the total current "enclosed" by the loop.

看i右手为正

Notes

- The *i's sign: obey right-hand rule* "+"
  not obey right-hand rule "-"

$$\oint \vec{B} \bullet d\vec{l} = \mu_0(+i_1 - i_2)$$

(2) The B at points on the loop and within loop certainly depends on the current i_3; however the integral of $\vec{B} \bullet d\vec{l}$ does not depends on the current i_3 that do not penetrate the surface by the loop.

积分只与里面电流有关

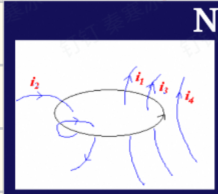
$$\oint_L Pdx + Qdy + Rdz$$

$$= \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \Sigma i \Rightarrow 求 B.$$

Notes (Cont.)

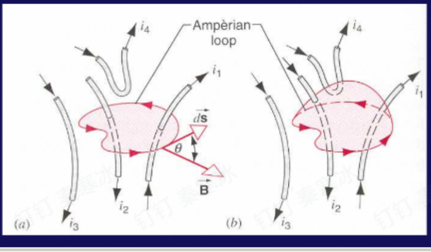


$$\oint \vec{B} \bullet d\vec{l} = \mu_0(i_1 + i_3 - 2i_2)$$

Due to $\frac{\mu_0}{4\pi}$ in Biot-Savart Law,
There is only a μ_0 in Ampere's Loop Law

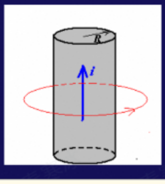
Stokes 公式

$$\oint \vec{B} \bullet d\vec{l} = \mu_0(i_1 - i_2)$$



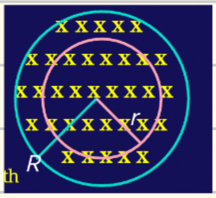
eg.

Example 1: Infinite long wire,
Radius of wire *R*;
i* : *uniform distribution



$$\oint \vec{B} \cdot d\vec{l} = B \cdot 2\pi r$$

$$B = \frac{\mu_0 i}{2\pi r}$$



内部半径r处 $\oint \vec{B} \cdot d\vec{l} = B \cdot 2\pi r$

$$i' = \frac{i}{\pi R^2} \cdot \pi r^2 = \frac{i r^2}{R^2} \Rightarrow B = \frac{\mu_0 i r}{2\pi R^2}$$

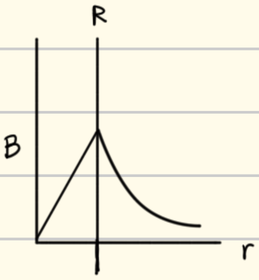
- **Inside the wire: (*r* < *R*)**

$$B = \frac{\mu_0 i r}{2\pi R^2}$$

B

- **Outside the wire: (*r*>*R*)**

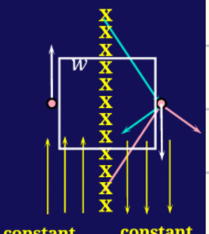
$$B = \frac{\mu_0 i}{2\pi r}$$



有线长的线，无法从安培环路出发求B！

Example 2: B Field of ∞ Current Sheet

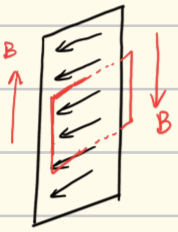
- Consider an ∞ sheet of current described by *n* wires/length each carrying current *i* into the screen as shown. Calculate the B field.
- What is the direction of the field?
  - Symmetry Þ vertical direction
- Calculate using Ampere's law for a square of side *w*:



$$\oint \vec{B} \bullet d\vec{l} = Bw + 0 + Bw + 0 = 2Bw = \mu_0 nwi$$

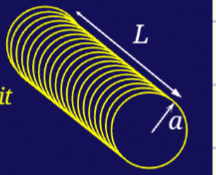
$$\longrightarrow \quad B = \frac{1}{2}\mu_0 ni$$

单位长度的电流线数



Example 3 B Field of a Solenoid

- A constant magnetic field can (in principle) be produced by an ¥ sheet of current. In practice, however, a constant magnetic field is often produced by a solenoid.
- A solenoid is defined by a current *i* flowing through a wire that is wrapped *n turns per unit length* on a cylinder of radius *a* and length *L*.



- If *a* << *L*, the B field is to first order contained within the solenoid, in the axial direction, and of constant magnitude. In this limit, we can calculate the field using Ampere's Law.

B Field of an ∞ Solenoid

- To calculate the B field of the ∞ solenoid using Ampere's Law, we need to justify the claim that the *B* field is 0 outside the solenoid.
- To do this, view the ∞ solenoid from the side as 2∞ current sheets.

- The fields are in the same direction in the region between the sheets (inside the solenoid) and cancel outside the sheets (outside the solenoid).
- **Draw square path of side w:**

$$\oint \vec{B} \bullet d\vec{l} = Bw = \mu_0 nwi$$

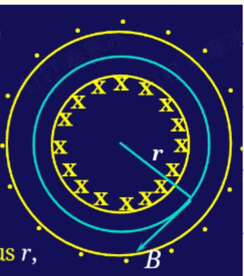
$$\longrightarrow \quad B = \mu_0 ni$$

Note: $B \propto \frac{\text{Amp}}{\text{Length}}$ ☑

Example 4 Toroid (螺绕环)

Toroid defined by *N total* turns with current *i*.

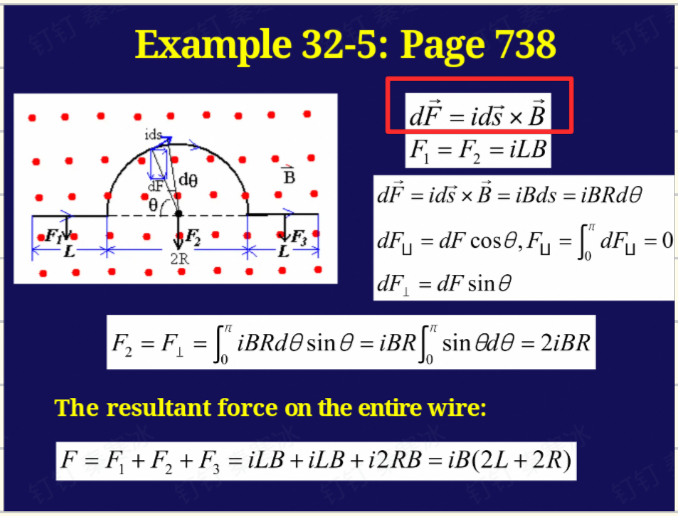
B=0 outside toroid! (Consider integrating *B* on circle outside toroid)



To find *B* inside, consider circle of radius *r*, centered at the center of the toroid.

$$\oint \vec{B} \bullet d\vec{l} = B \bullet 2\pi r = \mu_0 Ni$$

$$B = \frac{\mu_0 Ni}{2\pi r} = \mu_0 ni$$



Example 32-5: Page 738

$$\boxed{d\vec{F} = id\vec{s} \times \vec{B}}$$

$$F_1 = F_2 = iLB$$

$$d\vec{F} = id\vec{s} \times \vec{B} = iBds = iBRd\theta$$

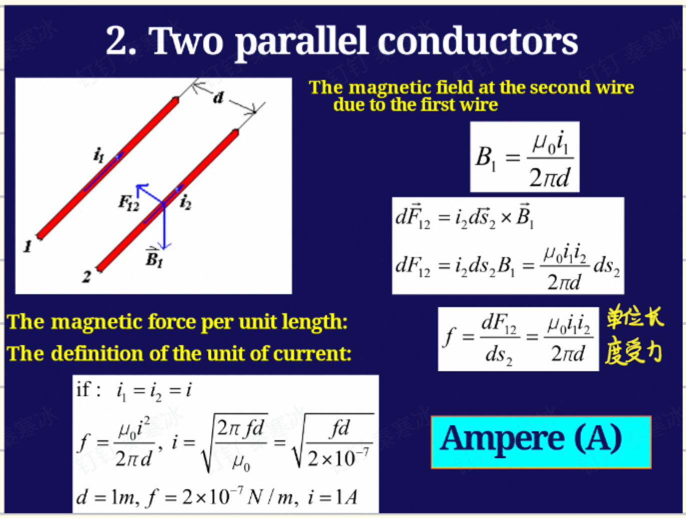
$$dF_{\sqcup} = dF\cos\theta, F_{\sqcup} = \int_0^{\pi} dF_{\sqcup} = 0$$

$$dF_{\perp} = dF\sin\theta$$

$$F_2 = F_{\perp} = \int_0^{\pi} iBRd\theta\sin\theta = iBR\int_0^{\pi}\sin\theta d\theta = 2iBR$$

The resultant force on the entire wire:

$$F = F_1 + F_2 + F_3 = iLB + iLB + i2RB = iB(2L + 2R)$$



2. Two parallel conductors

The magnetic field at the second wire due to the first wire

$$B_1 = \frac{\mu_0 i_1}{2\pi d}$$

$$d\vec{F}_{12} = i_2 d\vec{s}_2 \times \vec{B}_1$$

$$dF_{12} = i_2 ds_2 B_1 = \frac{\mu_0 i_1 i_2}{2\pi d} ds_2$$

The magnetic force per unit length:

$$f = \frac{dF_{12}}{ds_2} = \frac{\mu_0 i_1 i_2}{2\pi d}$$

单位长度受力

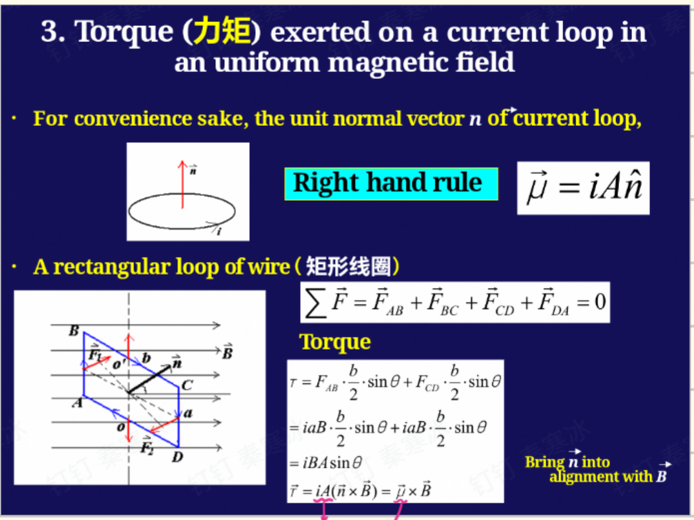
The definition of the unit of current:

$$\text{if}: \quad i_1 = i_2 = i$$

$$f = \frac{\mu_0 i^2}{2\pi d}, \quad i = \sqrt{\frac{2\pi f d}{\mu_0}} = \sqrt{\frac{fd}{2\times10^{-7}}}$$

$$d = 1m, \; f = 2\times10^{-7} N/m, \; i = 1A$$

Ampere (A)



3. Torque (力矩) exerted on a current loop in an uniform magnetic field

- For convenience sake, the unit normal vector n of current loop,

Right hand rule

$$\vec{\mu} = iA\hat{n}$$

- A rectangular loop of wire (矩形线圈)

$$\sum\vec{F} = \vec{F}_{AB} + \vec{F}_{BC} + \vec{F}_{CD} + \vec{F}_{DA} = 0$$

Torque

$$\tau = F_{AB}\cdot\frac{b}{2}\cdot\sin\theta + F_{CD}\cdot\frac{b}{2}\cdot\sin\theta$$

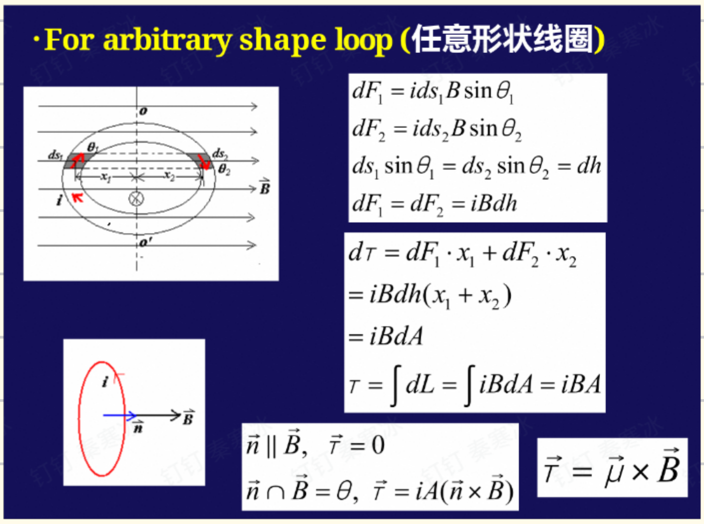
$$= iaB\cdot\frac{b}{2}\cdot\sin\theta + iaB\cdot\frac{b}{2}\cdot\sin\theta$$

$$= iBA\sin\theta$$

$$\vec{\tau} = \underbrace{iA}_{\text{磁偶极矩}}(\vec{n}\times\vec{B}) = \vec{\mu}\times\vec{B}$$

Bring $\vec{n}$ into alignment with $\vec{B}$

力矩方向: 顺时针转 (向里) 逆时针转 (向外)



·For arbitrary shape loop (任意形状线圈)

$$dF_1 = ids_1 B\sin\theta_1$$

$$dF_2 = ids_2 B\sin\theta_2$$

$$ds_1\sin\theta_1 = ds_2\sin\theta_2 = dh$$

$$dF_1 = dF_2 = iBdh$$

$$d\tau = dF_1\cdot x_1 + dF_2\cdot x_2$$

$$= iBdh(x_1 + x_2)$$

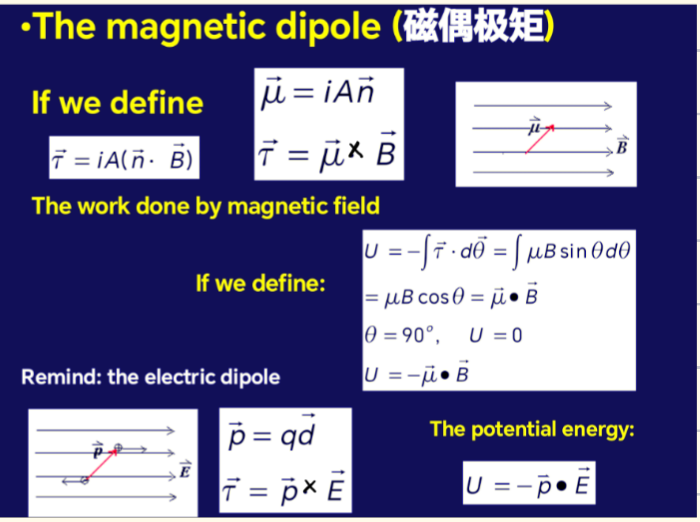
$$= iBdA$$

$$\tau = \int dL = \int iBdA = iBA$$

$$\vec{n}\parallel\vec{B}, \quad \vec{\tau} = 0$$

$$\vec{n}\cap\vec{B} = \theta, \; \vec{\tau} = iA(\vec{n}\times\vec{B})$$

$$\vec{\tau} = \vec{\mu}\times\vec{B}$$



•The magnetic dipole (磁偶极矩)

If we define

$$\vec{\tau} = iA(\vec{n}\cdot\;\vec{B})$$

$$\vec{\mu} = iA\vec{n}$$

$$\vec{\tau} = \vec{\mu}\times\vec{B}$$

The work done by magnetic field

If we define:

$$U = -\int\vec{\tau}\cdot d\vec{\theta} = \int\mu B\sin\theta d\theta$$

$$= \mu B\cos\theta = \vec{\mu}\bullet\vec{B}$$

$$\theta = 90^{\circ}, \quad U = 0$$

$$U = -\vec{\mu}\bullet\vec{B}$$

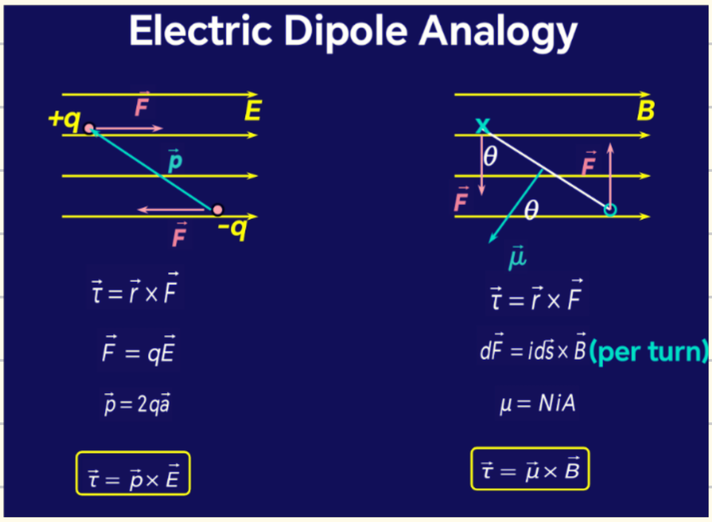
Remind: the electric dipole

$$\vec{p} = q\vec{d}$$

$$\vec{\tau} = \vec{p}\times\vec{E}$$

The potential energy:

$$U = -\vec{p}\bullet\vec{E}$$



Electric Dipole Analogy

$$\vec{\tau} = \vec{r}\times\vec{F}$$

$$\vec{F} = q\vec{E}$$

$$\vec{p} = 2q\vec{a}$$

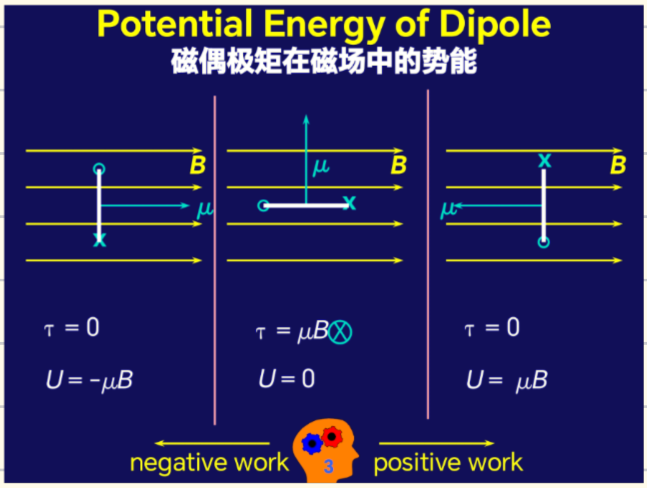
$$\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$$

$$\vec{\tau} = \vec{r}\times\vec{F}$$

$d\vec{F} = id\vec{s}\times\vec{B}$ (per turn)

$$\mu = NiA$$

$$\boxed{\vec{\tau} = \vec{\mu}\times\vec{B}}$$



Potential Energy of Dipole

磁偶极矩在磁场中的势能

$\tau = 0$

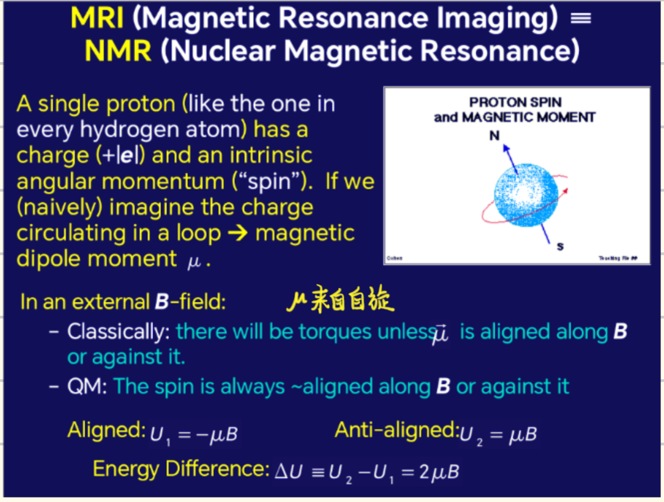
$U = -\mu B$

$\tau = \mu B \otimes$

$U = 0$

$\tau = 0$

$U = \mu B$



MRI (Magnetic Resonance Imaging) ≡ NMR (Nuclear Magnetic Resonance)

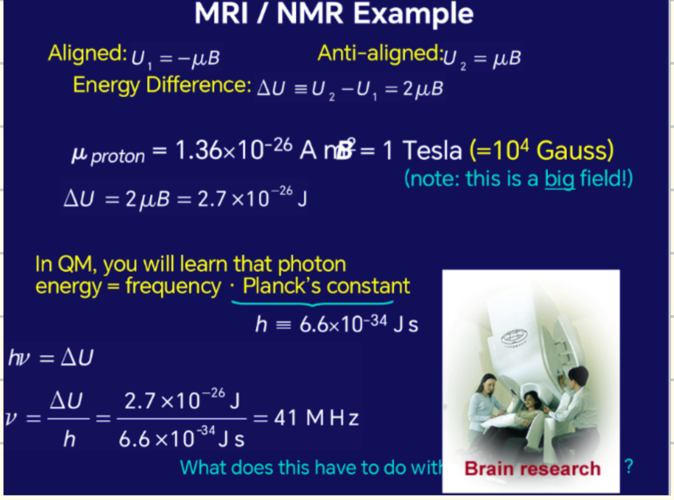
A single proton (like the one in every hydrogen atom) has a charge (+|e|) and an intrinsic angular momentum ("spin"). If we (naively) imagine the charge circulating in a loop → magnetic dipole moment μ.

In an external **B**-field: μ来自自旋

- Classically: there will be torques unless $\vec{\mu}$ is aligned along **B** or against it.
- QM: The spin is always ~aligned along **B** or against it

Aligned: $U_1 = -\mu B$ Anti-aligned: $U_2 = \mu B$

Energy Difference: $\Delta U \equiv U_2 - U_1 = 2\mu B$



MRI / NMR Example

Aligned: $U_1 = -\mu B$ Anti-aligned: $U_2 = \mu B$

Energy Difference: $\Delta U \equiv U_2 - U_1 = 2\mu B$

$\mu_{proton} = 1.36\times10^{-26}$ A m², B = 1 Tesla (=10^4 Gauss)

(note: this is a big field!)

$\Delta U = 2\mu B = 2.7\times10^{-26}$ J

In QM, you will learn that photon energy = frequency · Planck's constant

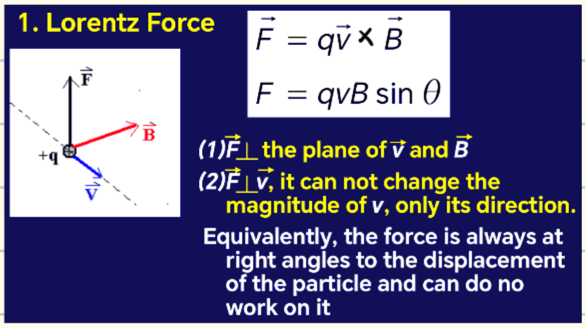
$h \equiv 6.6\times10^{-34}$ J s

$h\nu = \Delta U$

$$\nu = \frac{\Delta U}{h} = \frac{2.7\times10^{-26}\ \mathrm{J}}{6.6\times10^{-34}\ \mathrm{J\,s}} = 41\ \mathrm{MHz}$$

What does this have to do with Brain research ?

• 电荷在磁场中运动 (洛仑兹力)



1. Lorentz Force

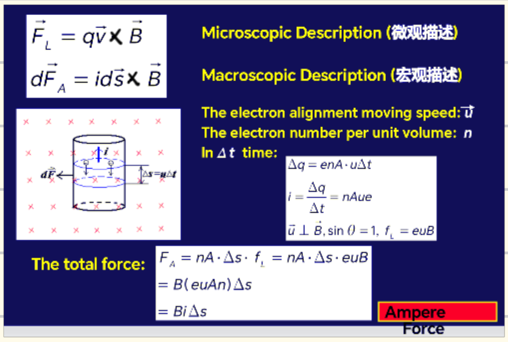
$$\vec{F} = q\vec{v}\times\vec{B}$$

$$F = qvB\sin\theta$$

(1) $\vec{F}\perp$ the plane of $\vec{v}$ and $\vec{B}$

(2) $\vec{F}\perp\vec{v}$, it can not change the magnitude of v, only its direction.

Equivalently, the force is always at right angles to the displacement of the particle and can do no work on it



$$\vec{F}_L = q\vec{v}\times\vec{B}$$

$$d\vec{F}_A = id\vec{s}\times\vec{B}$$

Microscopic Description (微观描述)

Macroscopic Description (宏观描述)

The electron alignment moving speed: $\vec{u}$

The electron number per unit volume: n

In Δt time:

$$\Delta q = enA\cdot u\Delta t$$

$$i = \frac{\Delta q}{\Delta t} = nAue$$

$$\vec{u}\perp\vec{B},\ \sin\theta = 1,\ f_L = euB$$

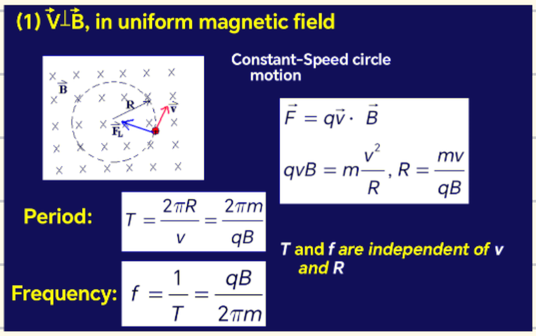
The total force:

$$F_A = nA\cdot\Delta s\cdot f_L = nA\cdot\Delta s\cdot euB$$

$$= B(euAn)\Delta s$$

$$= Bi\Delta s$$

Ampere Force



(1) $\vec{V}\perp\vec{B}$, in uniform magnetic field

Constant-Speed circle motion

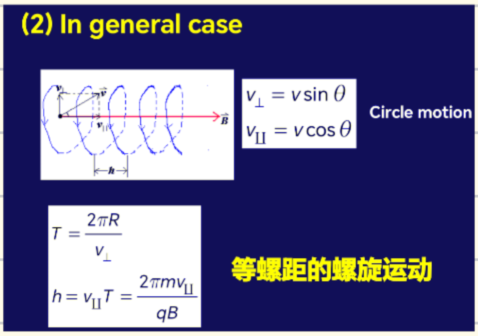
$$\vec{F} = q\vec{v}\cdot\vec{B}$$

$$qvB = m\frac{v^2}{R},\ R = \frac{mv}{qB}$$

Period: $T = \frac{2\pi R}{v} = \frac{2\pi m}{qB}$

Frequency: $f = \frac{1}{T} = \frac{qB}{2\pi m}$

T* and *f* are independent of *v* and *R



(2) In general case

$$v_\perp = v\sin\theta$$

$$v_{II} = v\cos\theta$$

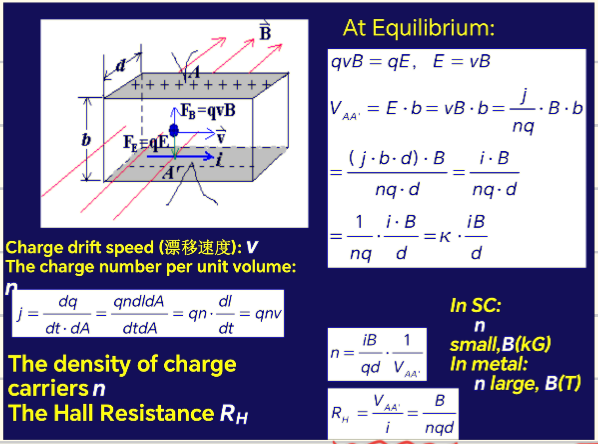
Circle motion

$$T = \frac{2\pi R}{v_\perp}$$

$$h = v_{II}T = \frac{2\pi m v_{II}}{qB}$$

等螺距的螺旋运动

• 霍尔效应



At Equilibrium:

$$qvB = qE,\quad E = vB$$

$$V_{AA'} = E\cdot b = vB\cdot b = \frac{j}{nq}\cdot B\cdot b$$

$$= \frac{(j\cdot b\cdot d)\cdot B}{nq\cdot d} = \frac{i\cdot B}{nq\cdot d}$$

$$= \frac{1}{nq}\cdot\frac{i\cdot B}{d} = \kappa\cdot\frac{iB}{d}$$

Charge drift speed (漂移速度): *V*

The charge number per unit volume: *n*

$$j = \frac{dq}{dt\cdot dA} = \frac{qndldA}{dtdA} = qn\cdot\frac{dl}{dt} = qnv$$

The density of charge carriers *n*

The Hall Resistance R_H

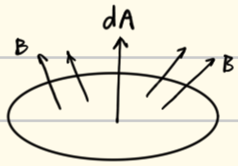
$$n = \frac{iB}{qd}\cdot\frac{1}{V_{AA'}}$$

$$R_H = \frac{V_{AA'}}{i} = \frac{B}{nqd}$$

In SC:
n small, B(kG)
In metal:
n large, B(T)

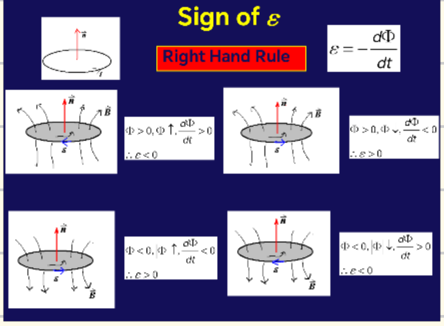
霍尔电阻

·法拉第电磁感应定律



$$\Phi_B=\iint\vec{B}\cdot d\vec{A}$$

$$\varepsilon=-\frac{d\Phi_B}{dt}$$



Sign of ε

Right Hand Rule

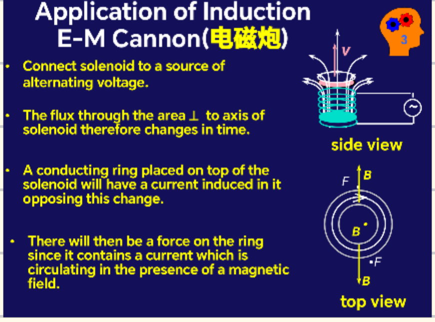
$$\varepsilon=-\frac{d\Phi}{dt}$$

$\Phi>0,\Phi\uparrow,\frac{d\Phi}{dt}>0$ $\therefore\varepsilon<0$

$\Phi>0,\Phi\downarrow,\frac{d\Phi}{dt}<0$ $\therefore\varepsilon>0$

$\Phi<0,|\Phi|\uparrow,\frac{d\Phi}{dt}<0$ $\therefore\varepsilon>0$

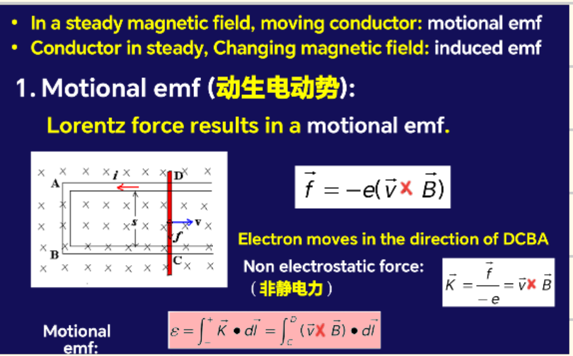
$\Phi<0,|\Phi|\downarrow,\frac{d\Phi}{dt}>0$ $\therefore\varepsilon<0$



Application of Induction E-M Cannon(电磁炮)

- Connect solenoid to a source of alternating voltage.
- The flux through the area ⊥ to axis of solenoid therefore changes in time.
- A conducting ring placed on top of the solenoid will have a current induced in it opposing this change.
- There will then be a force on the ring since it contains a current which is circulating in the presence of a magnetic field.

负号实际是楞次定律



- In a steady magnetic field, moving conductor: motional emf
- Conductor in steady, Changing magnetic field: induced emf

1. Motional emf (动生电动势):

Lorentz force results in a motional emf.

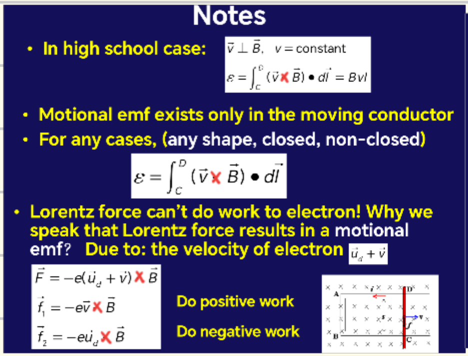
$$\vec{f}=-e(\vec{v}\times\vec{B})$$

Electron moves in the direction of DCBA

Non electrostatic force:（非静电力）

$$\vec{K}=\frac{\vec{f}}{-e}=\vec{v}\times\vec{B}$$

Motional emf: $\varepsilon=\int_{-}^{+}\vec{K}\bullet d\vec{l}=\int_{C}^{D}(\vec{v}\times\vec{B})\bullet d\vec{l}$



Notes

- In high school case: $\vec{v}\perp\vec{B},\quad v=\text{constant}$ $\varepsilon=\int_{C}^{D}(\vec{v}\times\vec{B})\bullet d\vec{l}=Bvl$
- Motional emf exists only in the moving conductor
- For any cases, (any shape, closed, non-closed)

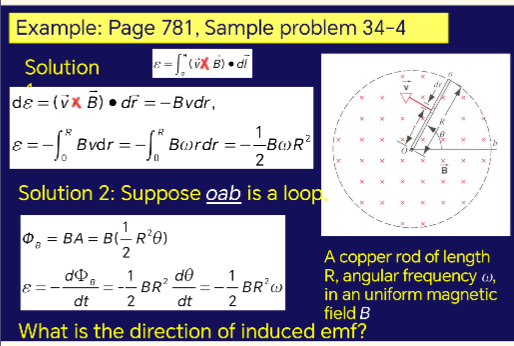
$$\varepsilon=\int_{C}^{D}(\vec{v}\times\vec{B})\bullet d\vec{l}$$

- Lorentz force can't do work to electron! Why we speak that Lorentz force results in a motional emf? Due to: the velocity of electron $\vec{u}_d+\vec{v}$

$\vec{F}=-e(\vec{u}_d+\vec{v})\times\vec{B}$

$\vec{f}_1=-e\vec{v}\times\vec{B}$ Do positive work

$\vec{f}_2=-e\vec{u}_d\times\vec{B}$ Do negative work



Example: Page 781, Sample problem 34-4

Solution 1 $\varepsilon=\int_{o}^{a}(\vec{v}\times\vec{B})\bullet d\vec{l}$

$$d\varepsilon=(\vec{v}\times\vec{B})\bullet d\vec{r}=-Bvdr,$$

$$\varepsilon=-\int_0^R Bvdr=-\int_0^R B\omega rdr=-\frac{1}{2}B\omega R^2$$

Solution 2: Suppose oab is a loop,

$$\Phi_B=BA=B(\frac{1}{2}R^2\theta)$$

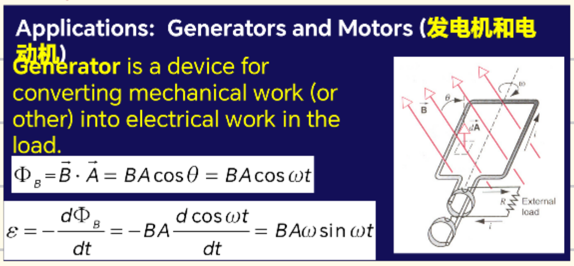
$$\varepsilon=-\frac{d\Phi_B}{dt}=-\frac{1}{2}BR^2\frac{d\theta}{dt}=-\frac{1}{2}BR^2\omega$$

A copper rod of length R, angular frequency ω, in an uniform magnetic field B

What is the direction of induced emf?

电动势ε

是非静电力将单位正电荷从负极移到正极所做的功

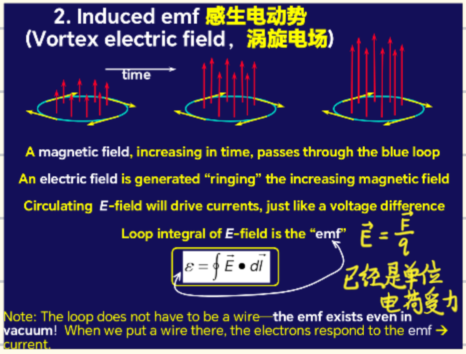


Applications: Generators and Motors (发电机和电动机)

Generator is a device for converting mechanical work (or other) into electrical work in the load.

$$\Phi_B=\vec{B}\cdot\vec{A}=BA\cos\theta=BA\cos\omega t$$

$$\varepsilon=-\frac{d\Phi_B}{dt}=-BA\frac{d\cos\omega t}{dt}=BA\omega\sin\omega t$$



2. Induced emf 感生电动势 (Vortex electric field，涡旋电场)

A magnetic field, increasing in time, passes through the blue loop

An electric field is generated "ringing" the increasing magnetic field

Circulating E-field will drive currents, just like a voltage difference

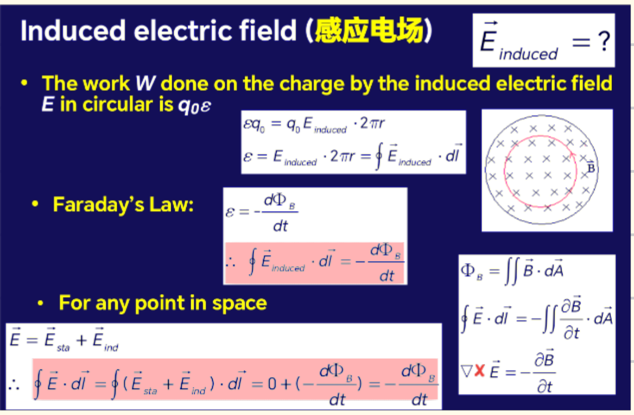
Loop integral of E-field is the "emf" $\vec{E}=\frac{\vec{F}}{q}$

$$\varepsilon=\oint\vec{E}\bullet d\vec{l}$$

已经是单位电荷受力

Note: The loop does not have to be a wire—the emf exists even in vacuum! When we put a wire there, the electrons respond to the emf → current.

变化的磁场产生电场



Induced electric field (感应电场)

$$\vec{E}_{induced}=?$$

- The work W done on the charge by the induced electric field E in circular is $q_0\varepsilon$

$$\varepsilon q_0=q_0E_{induced}\cdot 2\pi r$$

$$\varepsilon=E_{induced}\cdot 2\pi r=\oint\vec{E}_{induced}\cdot d\vec{l}$$

- Faraday's Law:

$$\varepsilon=-\frac{d\Phi_B}{dt}$$

$$\therefore\oint\vec{E}_{induced}\cdot d\vec{l}=-\frac{d\Phi_B}{dt}$$

- For any point in space

$$\vec{E}=\vec{E}_{sta}+\vec{E}_{ind}$$

$$\therefore\oint\vec{E}\cdot d\vec{l}=\oint(\vec{E}_{sta}+\vec{E}_{ind})\cdot d\vec{l}=0+(-\frac{d\Phi_B}{dt})=-\frac{d\Phi_B}{dt}$$

$$\Phi_B=\iint\vec{B}\cdot d\vec{A}$$

$$\oint\vec{E}\cdot d\vec{l}=-\iint\frac{\partial\vec{B}}{\partial t}\cdot d\vec{A}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

涡旋电场无电势差！

还是Stokes公式

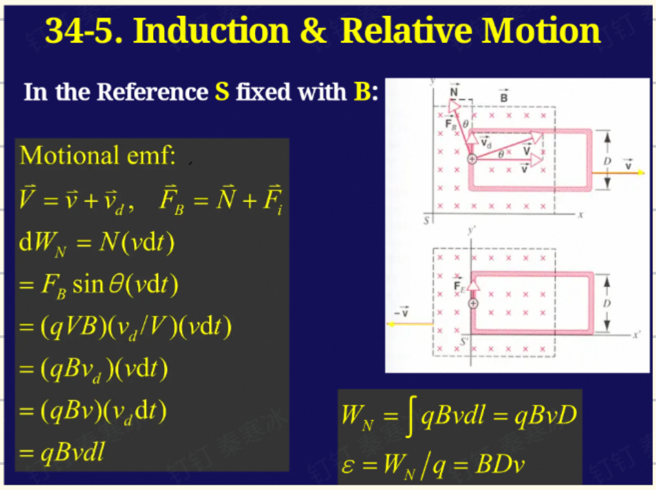
$$\oint\vec{E}\cdot d\vec{l}=\iint(\nabla\times\vec{E})\cdot d\vec{A}$$

$$\Downarrow$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

拓广的环路定理(静电场&感应电场)

闭合积分为0　闭合积分为电动势ε



34-5. Induction & Relative Motion

In the Reference S fixed with B:

Motional emf:

$$\vec{V}=\vec{v}+\vec{v}_d,\quad\vec{F}_B=\vec{N}+\vec{F}_i$$

$$dW_N=N(vdt)$$

$$=F_B\sin\theta(vdt)$$

$$=(qVB)(v_d/V)(vdt)$$

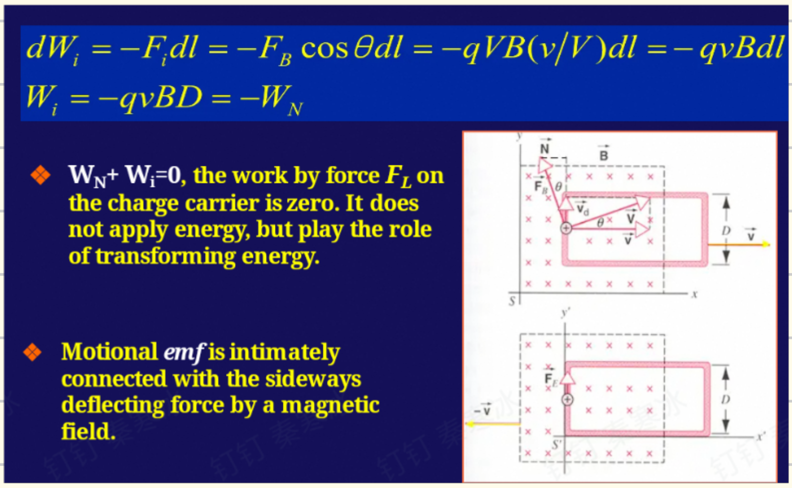
$$=(qBv_d)(vdt)$$

$$=(qBv)(v_ddt)$$

$$=qBvdl$$

$$W_N=\int qBvdl=qBvD$$

$$\varepsilon=W_N/q=BDv$$



$$dW_i=-F_idl=-F_B\cos\theta dl=-qVB(v/V)dl=-qvBdl$$

$$W_i=-qvBD=-W_N$$

- W_N+ W_i=0, the work by force F_L on the charge carrier is zero. It does not apply energy, but play the role of transforming energy.
- Motional *emf* is intimately connected with the sideways deflecting force by a magnetic field.

The Reference *S'* fixed with the loop:

$$\varepsilon' = \int E'dl = E'D$$

$$\because \varepsilon' = BDv$$

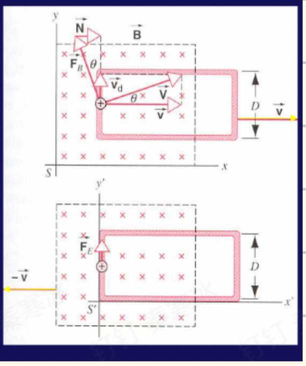
$$\vec{E}' = \vec{v} \times \vec{B}$$

Force is of purely electric origin,

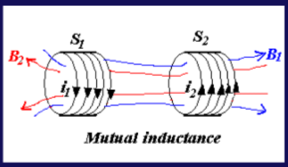
$$\varepsilon = \int \vec{E}' \cdot d\vec{l}$$

In general in S'':

$$\boxed{\varepsilon = \int (\vec{E}' + \vec{v} \times \vec{B}) \cdot d\vec{l}}$$



1. Mutual Inductance (互感）



From Faraday's Law

i_1 change s_2 induced emf ε_2

i_2 change s_1 induced emf ε_1

Mutual inductance emf ε_1 ε_2

The number of flux linkage (磁通匝链数) in S_2 due to S_1:

$$\Psi_{12} (\propto N_2 A_2 B_1) \propto N_2 \Phi_{12}$$
$$\Psi_{12} = M_{12} i_1$$

The number of flux linkage (磁通匝链数) in S_1 due to S_2:

$$\Psi_{21} (\propto N_1 A_1 B_2) \propto N_1 \Phi_{21}$$
$$\Psi_{21} = M_{21} i_2$$

From Faraday's Law

$$M_{12} = \frac{\Psi_{12}}{i_1} = \frac{N_2 \Phi_{12}}{i_1}; \quad \varepsilon_2 = -\frac{d\Psi_{12}}{dt} = -M_{12} \frac{di_1}{dt}, \quad (i_1 \text{ change})$$

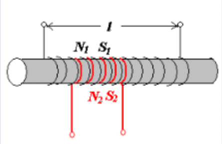
$$M_{21} = \frac{\Psi_{21}}{i_2} = \frac{N_1 \Phi_{21}}{i_2}; \quad \varepsilon_1 = -\frac{d\Psi_{21}}{dt} = -M_{21} \frac{di_2}{dt}, \quad (i_2 \text{ change})$$

Notes

- *M_{12}, M_{21} are called inductance constant* (互感系数）.
- *M_{12}=M_{21}=M*
- *Unit of M: Hery* (亨利）*H 1H = 1Wb/1A, mH, μH*

- **Example 1**

$$l = 1m, \ A = 10cm^2 = 10^{-3} m^2$$
$$N_1 = 1000, \quad N_2 = 20, \quad \frac{di_1}{dt} = 10 A/s$$

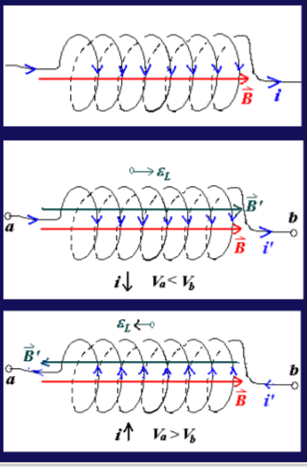


Calculate Mutual inductance,
induced emf ε_2 in S_2

$$B = \mu_0 n i, \quad B_1 = \mu_0 \frac{N_1}{l} i_1$$
$$\Psi_{12} = N_2 B_1 A = \mu_0 \frac{N_1 N_2 A}{l} i_1$$
$$M_{12} = \frac{\Psi_{12}}{i_1} = \mu_0 \frac{N_1 N_2 A}{l}$$
$$= 4\pi \cdot 10^{-7} \frac{1000 \cdot 20 \cdot 10^{-3}}{1}$$
$$= 25 \cdot 10^{-6} H = 25 \mu H$$

$$\varepsilon_2 = -M \frac{di_1}{dt}$$
$$= -25 \cdot 10^{-6} \cdot 10V$$
$$= -250 \mu V$$

2. Self-Inductance (自感)

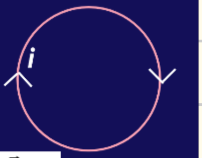


i change, $\vec{B}$ change
⟶ induced emf ε_L

$$\psi = NBA = Li$$
$$\varepsilon_L = -\frac{d\psi}{dt} = -L\frac{di}{dt}$$
$$V_b - V_a = -L\frac{di}{dt}$$

L ----- self-inductance

Self-Inductance *L*



- The magnetic field produced by the current in the loop shown is proportional to that current: $B \propto i$
- The flux, therefore, is also proportional to the current. $\Phi_B = \iint \vec{B} \bullet d\vec{A} \propto i$
- We define this constant of proportionality between flux and current to be the inductance, *L*. $\boxed{L = \frac{\Phi_B}{i}}$
- Combining with Faraday's Law gives the emf induced by a changing current:

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(Li) = -L\frac{di}{dt}$$

$$\boxed{\varepsilon = -L\frac{di}{dt}}$$

3. How to calculate the self-inductance(自感系数)

Similar to calculating the capacitance of a capacitor

suppose, q

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}, \quad (\text{Gauss'Law})$$
$$V = \int_a^b \vec{E} \bullet d\vec{l}$$
$$C = \frac{q}{V}$$

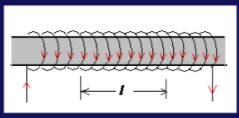
Calculate *L*:

- Suppose *i* in a particular inductor
- Determine $\vec{B}$
- The number of flux linkages:

$$\psi = N\Phi_B$$
$$= NBA$$
$$= Li$$

$$L = \frac{\Psi}{i} = \frac{N\Phi_B}{i}$$
$$\varepsilon_L = -\frac{d\Psi}{dt}$$
$$= -L\frac{di}{dt}$$
$$= -\frac{d}{dt}(N\Phi_B)$$

Example 2, The self-inductance of a solenoid



Calculate *L* for a section of length *l* of a long solenoid of cross-sectional area *A*

Suppose *i* in the solenoid.

$$B = \mu_0 ni$$

The number of flux linkages:

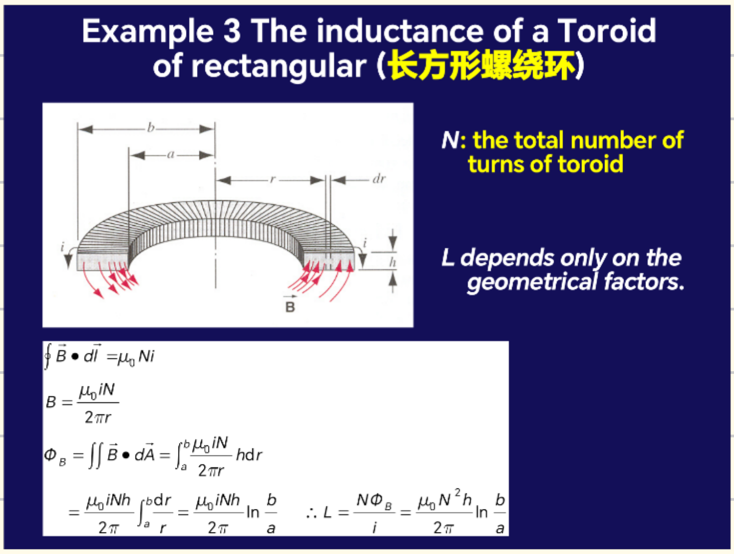
$$\psi = N\Phi_B = nlBA = \mu_0 n^2 ilA$$
$$L = \frac{\psi}{i} = \mu_0 n^2 lA = \mu_0 n^2 V$$

The self-inductance per unit volume:

$$L_v = \frac{L}{V} = \mu_0 n^2$$

The self-inductance per unit length:

$$L_v = \frac{L}{l} = \mu_0 n^2 A$$



Example 3 The inductance of a Toroid of rectangular (长方形螺绕环)

N: the total number of turns of toroid

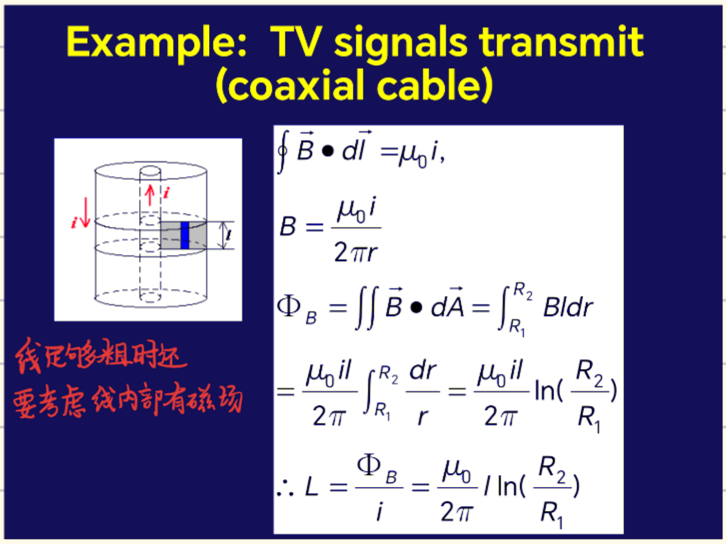
L depends only on the geometrical factors.

$$\oint \vec{B}\bullet d\vec{l} = \mu_0 Ni$$

$$B = \frac{\mu_0 iN}{2\pi r}$$

$$\Phi_B = \iint \vec{B}\bullet d\vec{A} = \int_a^b \frac{\mu_0 iN}{2\pi r} h\mathrm{d}r$$

$$= \frac{\mu_0 iNh}{2\pi}\int_a^b \frac{\mathrm{d}r}{r} = \frac{\mu_0 iNh}{2\pi}\ln\frac{b}{a} \qquad \therefore L = \frac{N\Phi_B}{i} = \frac{\mu_0 N^2 h}{2\pi}\ln\frac{b}{a}$$



Example: TV signals transmit (coaxial cable)

线足够粗时还
要考虑线内部有磁场

$$\oint \vec{B}\bullet d\vec{l} = \mu_0 i,$$

$$B = \frac{\mu_0 i}{2\pi r}$$

$$\Phi_B = \iint \vec{B}\bullet d\vec{A} = \int_{R_1}^{R_2} Bldr$$

$$= \frac{\mu_0 il}{2\pi}\int_{R_1}^{R_2}\frac{dr}{r} = \frac{\mu_0 il}{2\pi}\ln(\frac{R_2}{R_1})$$

$$\therefore L = \frac{\Phi_B}{i} = \frac{\mu_0}{2\pi} l\ln(\frac{R_2}{R_1})$$

一、自感系数（L）

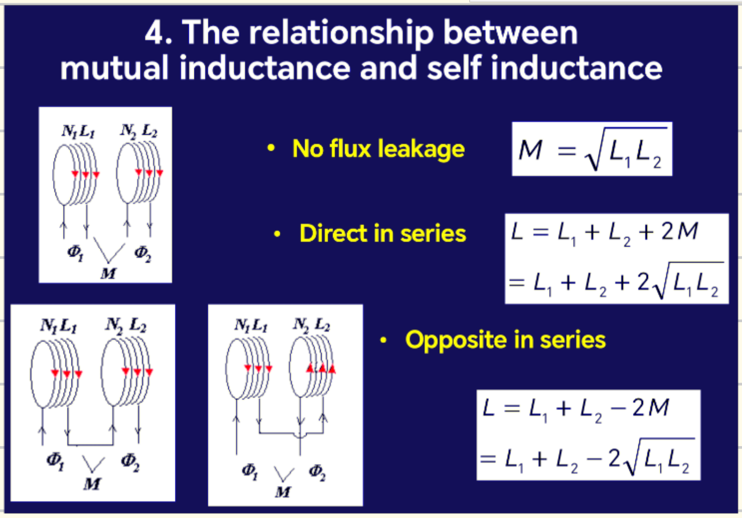
1. 定义

当一个线圈中的电流发生变化时，它产生的变化磁场会穿过自身线圈，从而在**自身**中感应出电动势（这种现象叫**自感现象**）。自感系数 L 是描述“线圈自身电流变化时，产生自感电动势能力”的物理量。

二、互感系数（M）

1. 定义

当两个线圈（如 S_1 和 S_2）之间存在磁场耦合时，一个线圈的电流变化会在**另一个**线圈中感应出电动势（这种现象叫**互感现象**）。互感系数 M 是描述“两个线圈之间磁场耦合强弱”的物理量。



4. The relationship between mutual inductance and self inductance

- No flux leakage

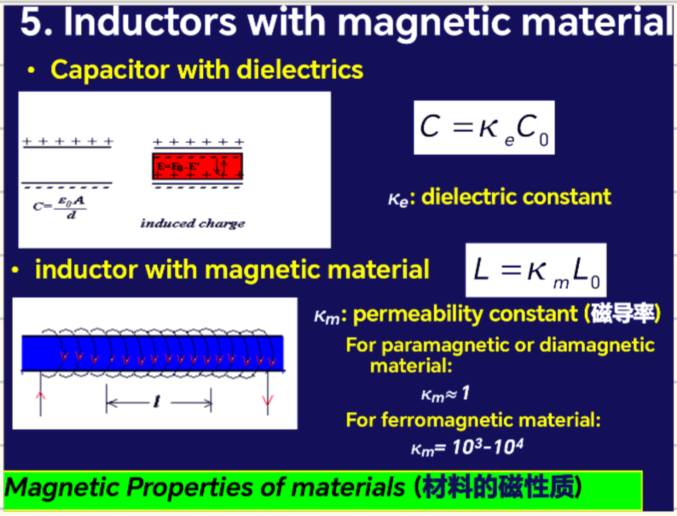
$$M = \sqrt{L_1 L_2}$$

- Direct in series

$$L = L_1 + L_2 + 2M$$
$$= L_1 + L_2 + 2\sqrt{L_1 L_2}$$

- Opposite in series

$$L = L_1 + L_2 - 2M$$
$$= L_1 + L_2 - 2\sqrt{L_1 L_2}$$



5. Inductors with magnetic material

- Capacitor with dielectrics

$$C = \kappa_e C_0$$

κ_e: dielectric constant

- inductor with magnetic material

$$L = \kappa_m L_0$$

κ_m: permeability constant (磁导率)

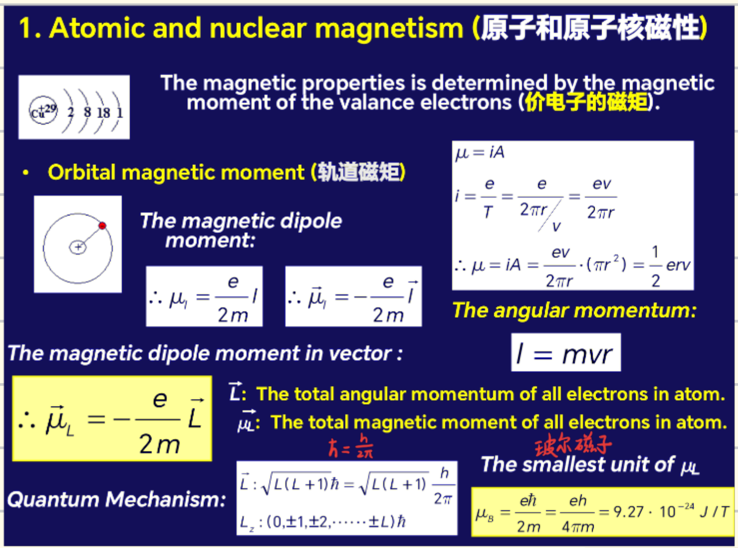
For paramagnetic or diamagnetic material:

$\kappa_m \approx 1$

For ferromagnetic material:

$\kappa_m = 10^3 - 10^4$

Magnetic Properties of materials (材料的磁性质)



1. Atomic and nuclear magnetism (原子和原子核磁性)

The magnetic properties is determined by the magnetic moment of the valance electrons (价电子的磁矩).

- Orbital magnetic moment (轨道磁矩)

The magnetic dipole moment:

$$\mu = iA$$

$$i = \frac{e}{T} = \frac{e}{2\pi r/v} = \frac{ev}{2\pi r}$$

$$\therefore \mu = iA = \frac{ev}{2\pi r}\cdot(\pi r^2) = \frac{1}{2}erv$$

$$\therefore \mu_l = \frac{e}{2m}l \qquad \therefore \vec{\mu}_l = -\frac{e}{2m}\vec{l}$$

The angular momentum:

$$l = mvr$$

The magnetic dipole moment in vector :

$$\therefore \vec{\mu}_L = -\frac{e}{2m}\vec{L}$$

$\vec{L}$: The total angular momentum of all electrons in atom.

$\vec{\mu_L}$: The total magnetic moment of all electrons in atom.

$\hbar = \frac{h}{2\pi}$

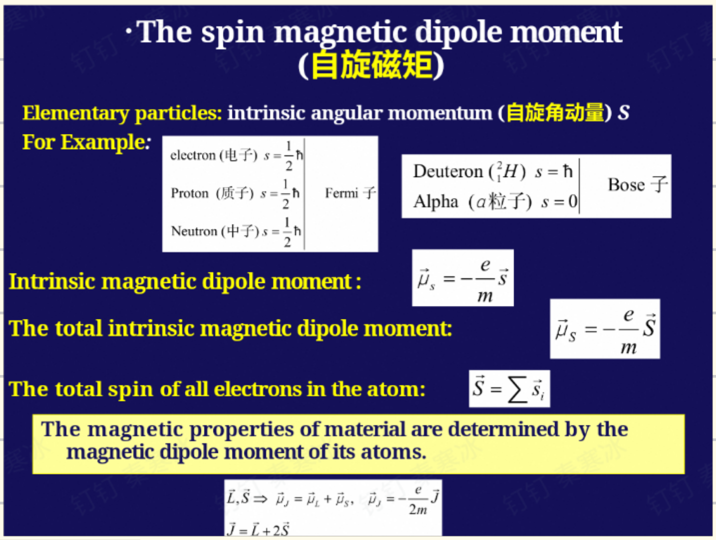
Quantum Mechanism:

$$\vec{L}: \sqrt{L(L+1)}\hbar = \sqrt{L(L+1)}\frac{h}{2\pi}$$
$$L_z: (0,\pm1,\pm2,\cdots\cdots\pm L)\hbar$$

玻尔磁子

The smallest unit of μ_L

$$\mu_B = \frac{e\hbar}{2m} = \frac{eh}{4\pi m} = 9.27\cdot 10^{-24}\ J/T$$



·The spin magnetic dipole moment (自旋磁矩)

Elementary particles: intrinsic angular momentum (自旋角动量) *S*

For Example:

electron (电子) $s=\frac{1}{2}\hbar$ Proton (质子) $s=\frac{1}{2}\hbar$ Neutron (中子) $s=\frac{1}{2}\hbar$	Fermi 子

Deuteron (^2_1H) $s=\hbar$ Alpha (α粒子) $s=0$	Bose 子

Intrinsic magnetic dipole moment : $\vec{\mu}_s = -\frac{e}{m}\vec{s}$

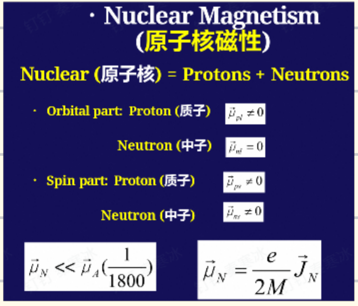
The total intrinsic magnetic dipole moment: $\vec{\mu}_S = -\frac{e}{m}\vec{S}$

The total spin of all electrons in the atom: $\vec{S} = \sum \vec{s}_i$

The magnetic properties of material are determined by the magnetic dipole moment of its atoms.

$$\vec{L},\vec{S} \Rightarrow \vec{\mu}_J = \vec{\mu}_L + \vec{\mu}_S,\quad \vec{\mu}_J = -\frac{e}{2m}\vec{J}$$
$$\vec{J} = \vec{L} + 2\vec{S}$$

原子磁性取决于自旋磁矩和轨道磁矩求和



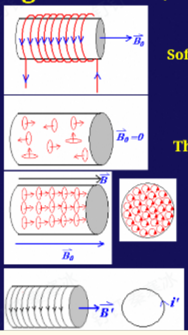
· Nuclear Magnetism (原子核磁性)

Nuclear (原子核) = Protons + Neutrons

- Orbital part: Proton (质子) $\vec{\mu}_{pl} \neq 0$
  Neutron (中子) $\vec{\mu}_{nl} = 0$
- Spin part: Proton (质子) $\vec{\mu}_{ps} \neq 0$
  Neutron (中子) $\vec{\mu}_{ns} \neq 0$

$$\vec{\mu}_N << \vec{\mu}_A(\frac{1}{1800}) \qquad \vec{\mu}_N = \frac{e}{2M}\vec{J}_N$$

2. Magnetization (磁化强度M) of material



Soft-Fe rod (软铁棒)

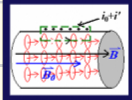
The macroscopic result (宏观结果):

$$\vec{\tau} = \vec{\mu}\times\vec{B}$$

$$\vec{B} = \vec{B}_0 + \vec{B}'_M$$

Induced Current
(束缚电流)

Magnetization Vector (磁化强度矢量M)



Define magnetization Vector

$$\vec{M} = \frac{\sum\vec{\mu}_m}{\Delta V}$$

$$\oint\vec{M}\bullet d\vec{l} = \sum i'$$ 束缚电流

$$\vec{M}\times\vec{n} = \vec{j}'$$

Induced current density
(单位长度电流，束缚电流密度)

Polarization:

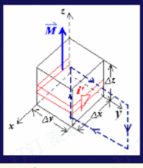
$$\vec{P} = \frac{\sum\vec{p}_m}{\Delta V}$$

$$\oiint\vec{P}\bullet d\vec{A} = -\sum q'$$

$$\vec{P}\bullet\vec{n} = \sigma'_e$$

$$\oint\vec{M}\bullet d\vec{l} = i'$$

Show:



Uniform magnetization (均匀磁化):

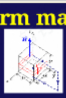
$$j' = \frac{i'}{\Delta z}$$

$$\Delta m = i'\cdot\Delta A = j'\Delta x\Delta y\Delta z$$

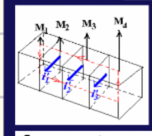
$$M = \frac{\Delta m}{\Delta V} = j'$$

$$M\cdot\Delta z = i'$$

Non-uniform magnetization (非均匀磁化):



$$M\cdot\Delta z = i'$$



$$i_1' = i_2 - i_1 = (M_2 - M_1)\bullet\Delta z$$

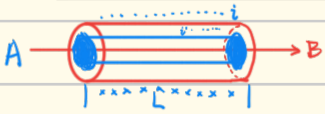
$$i_2' = i_3 - i_2 = (M_3 - M_2)\bullet\Delta z$$

$$i_3' = i_4 - i_3 = (M_4 - M_3)\bullet\Delta z$$

$$\therefore\quad i_1' + i_2' + i_3' = (M_4 - M_1)\bullet\Delta z$$

$$\oint\vec{M}\bullet d\vec{l} = M_4\cdot\Delta z - M_1\cdot\Delta z = i_1' + i_2' + i_3' = \sum_{inl} i'$$

$$\oint_l\vec{M}\bullet d\vec{l} = \sum_{inl} i'$$



$V = AL$ $\quad j_m\cdot L = \sum i'$

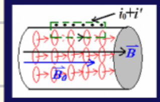
$\sum\mu_m = \sum i_m A$

$= \sum j_m LA = \sum j_m V$

$\therefore \sum j_m = \frac{\sum\mu_m}{V}$ $\quad\therefore \oint\overset{M}{\frac{\sum\mu_m}{\Delta V}}\cdot dl = \sum i'$

Ampere's Loop Law with magnetic material

The magnetic induction strength in the magnetic medium (磁性介质)



$$\vec{B} = \vec{B}_0 + \vec{B}_M$$

$$\oint_l\vec{M}\bullet d\vec{l} = \sum_{inl} i'$$

$$\oint_L\vec{B}\bullet d\vec{l} = \mu_0\sum_{inl}(i_0 + i') = \mu_0\sum_{inl} i_0 + \mu_0\oint_L\vec{M}\bullet d\vec{l}$$

$$\oint_L\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)\bullet d\vec{l} = \sum_{inL} i_0$$

Define magnetic field strength (磁场强度) $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

New Ampere's Loop Law $\oint_l\vec{H}\bullet d\vec{l} = \sum_{inl} i_0$

Gauss' Law $\oiint\vec{B}\bullet d\vec{A} = 0$

Because Biot-Savart Law for *i* and *i'* is always value.

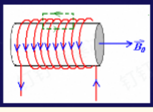
Notes for magnetic field strength $\vec{H}$

Define magnetic field strength (磁场强度) $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

- Unit of *H*: *Os*(奥斯特), *A/m*
  1A/m = 4π×10⁻³Os

$$\oint_l\vec{H}\bullet d\vec{l} = \sum_{inl} i_0$$

- In vacuum $\vec{M} = 0,\ \vec{H} = \frac{\vec{B}}{\mu_0},\ \vec{B} = \mu_0\vec{H}$
- For a solenoid with soft-Fe rod:



$$\oint\vec{H}\bullet d\vec{l} = \sum i_0$$

$$H\cdot\Delta l = Ni_0$$

$$H = ni_0$$

$$B_0 = \mu_0 ni_0$$

$$B_0 = \mu_0 H$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \vec{B}_0 + \mu_0\vec{M}$$

3. The magnetization Law for material

⟶ Relationship between $\vec{M}$, $\vec{B}$ and $\vec{H}$

For dielectrics:

$$\vec{P} = \chi_e\varepsilon_0\vec{E}$$

$$\vec{D} = \kappa_e\varepsilon_0\vec{E}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$= (1+\chi_e)\varepsilon_0\vec{E}$$

$$= \kappa_e\varepsilon_0\vec{E}$$

$$\kappa_e = 1 + \chi_e$$

For magnetic materials:

$$\vec{M} = \chi_m\vec{H}$$

$$\vec{B} = \kappa_m\mu_0\vec{H}$$

χ_m: susceptibility (磁化率)
magnetization coefficient

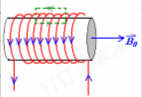
κ_m: permeability constant (磁导率)

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1+\chi_m)\vec{H} = \kappa_m\mu_0\vec{H}$$

$$\therefore\kappa_m = 1 + \chi_m$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

For a solenoid with soft-Fe rod:



$$\oint\vec{H}\bullet d\vec{l} = \sum i_0$$

$$H\cdot\Delta l = Ni_0$$

$$H = ni_0$$

$$B = \kappa_m\mu_0 H = \kappa_m\mu_0 ni_0 = \kappa_m B_0$$

$$\therefore\frac{B}{B_0} = \kappa_m$$

$$\frac{\psi}{\psi_0} = \kappa_m \Rightarrow \frac{L}{L_0} = \kappa_m$$

- **For dielectrics:** $\chi_e > 0, \kappa_e > 1$

$$\vec{M} = \chi_m\vec{H}$$

$$\vec{B} = \kappa_m\mu_0\vec{H}$$

- **For magnetic materials:**

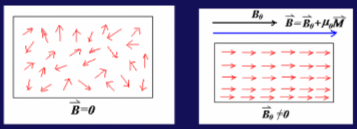
Paramagnetic (顺磁) materials: $\chi_m > 0, \kappa_m > 1, (\chi_m \approx 10^{-6}) \therefore \kappa_m \approx 1$

Diamagnetic (抗磁) materials: $\chi_m < 0, \kappa_m < 1, |\chi_m| << 1, \therefore \kappa_m \approx 1$

Ferromagnetic (铁磁) materials: $\chi_m(H), \kappa_m(H); (\kappa_m \approx 10^2 - 10^3)$

Microscopic Explanation (微观解释)

1. For paramagnetic materials (顺磁材料):



$\vec{B}=0$ $\quad\vec{B}_0\neq0$

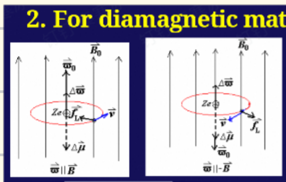
$$\vec{\mu}_m \neq 0$$

The temperature dependence

$$\vec{M} = \chi_m\vec{H} = \frac{C}{T}\vec{H}$$

$$\chi_m(T) = \frac{C}{T}$$ **Curie Law**

2. For diamagnetic materials (抗磁材料)



$$\vec{\mu}_m = 0,\ \vec{J} = 0$$

$$\frac{Ze^2}{4\pi\varepsilon_0 r^2} = m\omega_0^2 r$$

$$\omega_0 = \left(\frac{Ze^2}{4\pi\varepsilon_0 mr^3}\right)^{1/2}$$

$\vec{\omega}\parallel\vec{B}$

$$\frac{Ze^2}{4\pi\varepsilon_0 r^2} + e\omega rB = m\omega^2 r$$

$$\omega = \omega_0 + \Delta\omega$$

$$\Delta\omega = \frac{eB}{2m}$$

$\vec{\omega}\parallel-\vec{B}$

$$\frac{Ze^2}{4\pi\varepsilon_0 r^2} - e\omega rB = m\omega^2 r$$

$$\omega = \omega_0 - \Delta\omega$$

$$\Delta\omega = \frac{eB}{2m}$$

$$\mu = iA = \frac{ev}{2\pi r}(\pi r^2) = \frac{1}{2}evr = \frac{er^2}{2}\omega,\quad \vec{\mu}_0 = -\frac{er^2}{2}\vec{\omega}_0$$

$$\Delta\vec{\mu} = -\frac{er^2}{2}\Delta\vec{\omega} = -\frac{e^2r^2}{4m}\vec{B}$$

3.Ferromagnetic materials (铁磁材料)

$\vec{\mu}_m \neq 0$ + strong interaction between neighboring atomic dipole moments 近邻原子磁矩间存在强相互作用

- Fe, Co, Ni at room temperature
- Gd, Dy at low temperatures
- CrO_2 the magnetic powder on the tape.

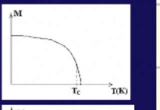
T ↑, Ferromagnetic ⟶ paramagnetic
the interaction ↓

Curie-Weiss Law

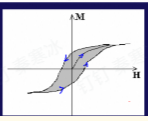
$$\chi_m = \frac{M}{H} = \frac{C}{T-\theta}$$

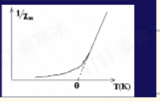
For Ferromagnets, $\theta > 0$

$Fe,\quad \theta = 770^\circ C$
$Gd,\quad \theta = 16^\circ C$



Magnetic hysteresis:
Domain(磁畴)





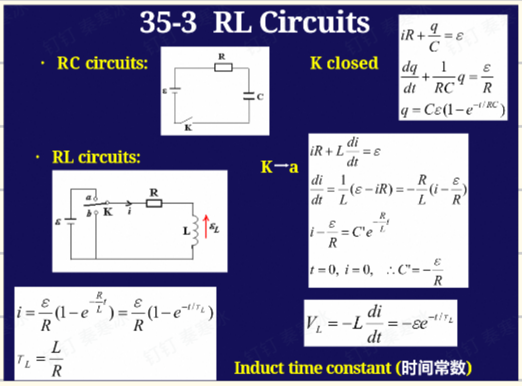
Ferromagnets, cont.

- Even in the absence of an applied B, the dipoles tend to strongly align over small patches – "domains (磁畴)". Applying an external field, the domains align to produce a large net magnetization.

Magnetic Domains

0.01*mm*

- "Soft" ferromagnets 软铁磁体
  - The domains re-randomize when the field is removed
- "Hard" ferromagnets 硬铁磁体
  - The domains persist even when the field is removed
  - "Permanent" magnets 永久磁体
    - Domains may be aligned in a different direction by applying a new field
    - Domains may be re-randomized by sudden physical shock
    - If the temperature is raised above the "Curie point" (770˚ for iron), the domains will also randomize paramagnet



35-3 RL Circuits

- **RC circuits:**

K closed

$$iR+\frac{q}{C}=\varepsilon$$
$$\frac{dq}{dt}+\frac{1}{RC}q=\frac{\varepsilon}{R}$$
$$q=C\varepsilon(1-e^{-t/RC})$$

- **RL circuits:**

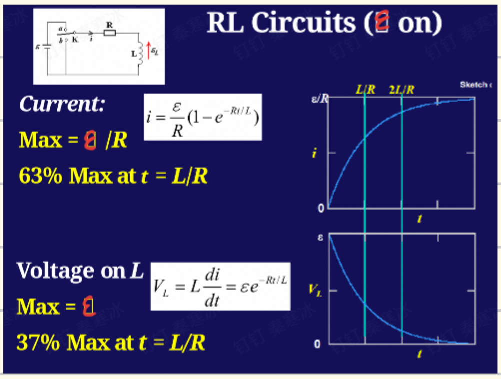
K→a

$$iR+L\frac{di}{dt}=\varepsilon$$
$$\frac{di}{dt}=\frac{1}{L}(\varepsilon-iR)=-\frac{R}{L}(i-\frac{\varepsilon}{R})$$
$$i-\frac{\varepsilon}{R}=C'e^{-\frac{R}{L}t}$$
$$t=0,\ i=0,\quad \therefore C'=-\frac{\varepsilon}{R}$$

$$i=\frac{\varepsilon}{R}(1-e^{-\frac{R}{L}t})=\frac{\varepsilon}{R}(1-e^{-t/\tau_L})$$
$$\tau_L=\frac{L}{R}$$

$$V_L=-L\frac{di}{dt}=-\varepsilon e^{-t/\tau_L}$$

Induct time constant (时间常数)



RL Circuits (ε on)

Current:

$$i=\frac{\varepsilon}{R}(1-e^{-Rt/L})$$

Max = ε /R

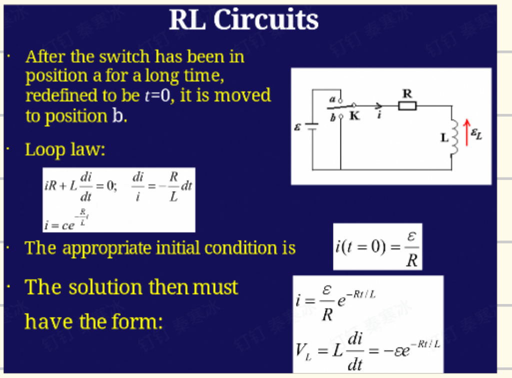
63% Max at *t* = *L*/*R*

Voltage on *L*

$$V_L=L\frac{di}{dt}=\varepsilon e^{-Rt/L}$$

Max = ε

37% Max at *t* = *L*/*R*



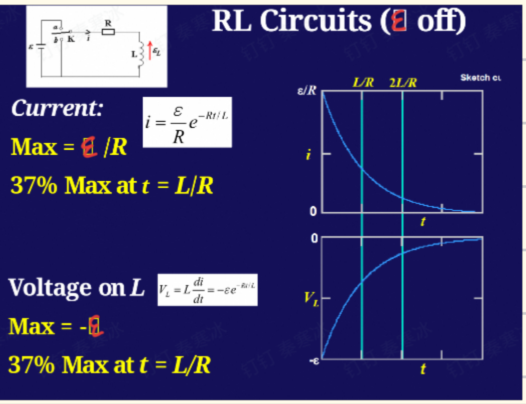
RL Circuits

- After the switch has been in position a for a long time, redefined to be *t*=0, it is moved to position b.
- Loop law:

$$iR+L\frac{di}{dt}=0;\quad \frac{di}{i}=-\frac{R}{L}dt$$
$$i=ce^{-\frac{R}{L}t}$$

- The appropriate initial condition is $i(t=0)=\frac{\varepsilon}{R}$
- The solution then must have the form:

$$i=\frac{\varepsilon}{R}e^{-Rt/L}$$
$$V_L=L\frac{di}{dt}=-\varepsilon e^{-Rt/L}$$



RL Circuits (ε off)

Current:

$$i=\frac{\varepsilon}{R}e^{-Rt/L}$$

Max = ε /R

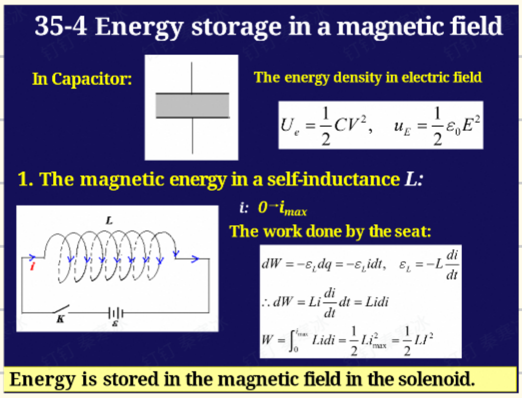
37% Max at *t* = *L*/*R*

Voltage on *L*

$$V_L=L\frac{di}{dt}=-\varepsilon e^{-Rt/L}$$

Max = -ε

37% Max at *t* = *L*/*R*



35-4 Energy storage in a magnetic field

In Capacitor:

The energy density in electric field

$$U_e=\frac{1}{2}CV^2,\quad u_E=\frac{1}{2}\varepsilon_0E^2$$

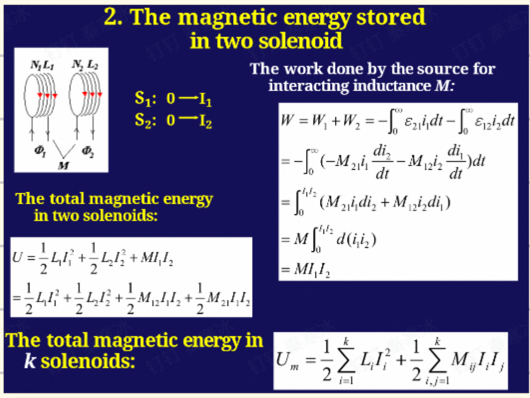
1. The magnetic energy in a self-inductance *L*:

i: *0→i_{max}*

The work done by the seat:

$$dW=-\varepsilon_Ldq=-\varepsilon_Lidt,\quad \varepsilon_L=-L\frac{di}{dt}$$
$$\therefore dW=Li\frac{di}{dt}dt=Lidi$$
$$W=\int_0^{i_{max}}Lidi=\frac{1}{2}Li_{max}^2=\frac{1}{2}LI^2$$

Energy is stored in the magnetic field in the solenoid.



2. The magnetic energy stored in two solenoid

S$_1$: 0⟶I$_1$
S$_2$: 0⟶I$_2$

The work done by the source for interacting inductance *M*:

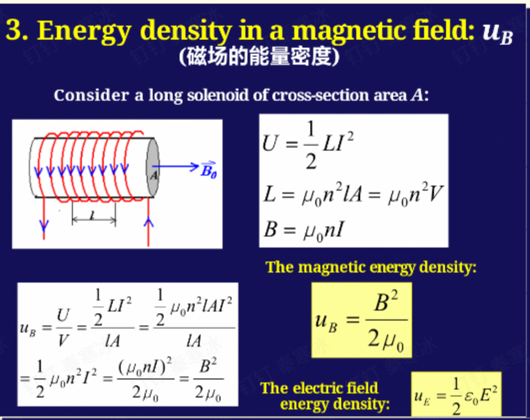
$$W=W_1+W_2=-\int_0^\infty\varepsilon_{21}i_1dt-\int_0^\infty\varepsilon_{12}i_2dt$$
$$=-\int_0^\infty(-M_{21}i_1\frac{di_2}{dt}-M_{12}i_2\frac{di_1}{dt})dt$$
$$=\int_0^{I_1I_2}(M_{21}i_1di_2+M_{12}i_2di_1)$$
$$=M\int_0^{I_1I_2}d(i_1i_2)$$
$$=MI_1I_2$$

The total magnetic energy in two solenoids:

$$U=\frac{1}{2}L_1I_1^2+\frac{1}{2}L_2I_2^2+MI_1I_2$$
$$=\frac{1}{2}L_1I_1^2+\frac{1}{2}L_2I_2^2+\frac{1}{2}M_{12}I_1I_2+\frac{1}{2}M_{21}I_1I_2$$

The total magnetic energy in *k* solenoids:

$$U_m=\frac{1}{2}\sum_{i=1}^{k}L_iI_i^2+\frac{1}{2}\sum_{i,j=1}^{k}M_{ij}I_iI_j$$



3. Energy density in a magnetic field: u_B

(磁场的能量密度)

Consider a long solenoid of cross-section area *A*:

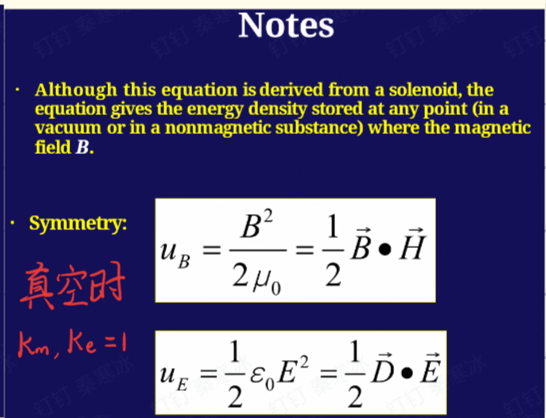
$$U=\frac{1}{2}LI^2$$
$$L=\mu_0n^2lA=\mu_0n^2V$$
$$B=\mu_0nI$$

The magnetic energy density:

$$u_B=\frac{U}{V}=\frac{\frac{1}{2}LI^2}{lA}=\frac{\frac{1}{2}\mu_0n^2lAI^2}{lA}$$
$$=\frac{1}{2}\mu_0n^2I^2=\frac{(\mu_0nI)^2}{2\mu_0}=\frac{B^2}{2\mu_0}$$

$$u_B=\frac{B^2}{2\mu_0}$$

The electric field energy density: $u_E=\frac{1}{2}\varepsilon_0E^2$



Notes

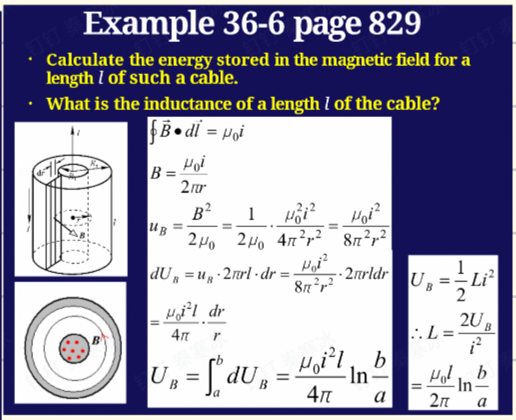
- **Although this equation is derived from a solenoid, the equation gives the energy density stored at any point (in a vacuum or in a nonmagnetic substance) where the magnetic field *B*.**
- **Symmetry:**

$$u_B=\frac{B^2}{2\mu_0}=\frac{1}{2}\vec{B}\bullet\vec{H}$$

$$u_E=\frac{1}{2}\varepsilon_0E^2=\frac{1}{2}\vec{D}\bullet\vec{E}$$

真空时

$K_m, K_e=1$

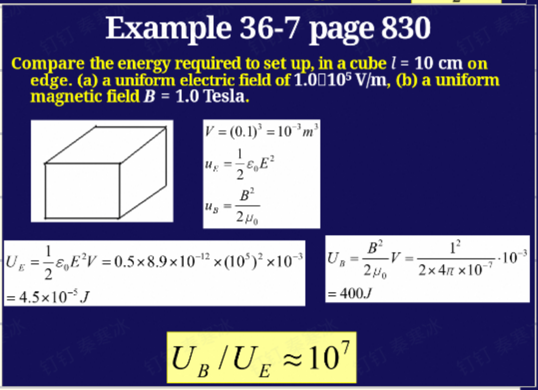


Example 36-6 page 829

- **Calculate the energy stored in the magnetic field for a length *l* of such a cable.**
- **What is the inductance of a length *l* of the cable?**

$$\oint\vec{B}\bullet d\vec{l}=\mu_0i$$
$$B=\frac{\mu_0i}{2\pi r}$$
$$u_B=\frac{B^2}{2\mu_0}=\frac{1}{2\mu_0}\cdot\frac{\mu_0^2i^2}{4\pi^2r^2}=\frac{\mu_0i^2}{8\pi^2r^2}$$
$$dU_B=u_B\cdot2\pi rl\cdot dr=\frac{\mu_0i^2}{8\pi^2r^2}\cdot2\pi rldr$$
$$=\frac{\mu_0i^2l}{4\pi}\cdot\frac{dr}{r}$$
$$U_B=\int_a^bdU_B=\frac{\mu_0i^2l}{4\pi}\ln\frac{b}{a}$$

$$U_B=\frac{1}{2}Li^2$$
$$\therefore L=\frac{2U_B}{i^2}$$
$$=\frac{\mu_0l}{2\pi}\ln\frac{b}{a}$$



Example 36-7 page 830

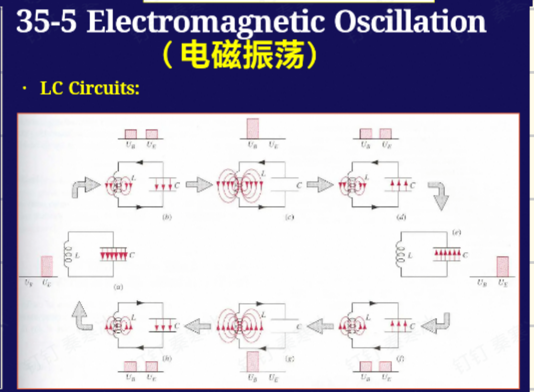
Compare the energy required to set up, in a cube *l* = 10 cm on edge. (a) a uniform electric field of 1.0×10^5 V/m, (b) a uniform magnetic field *B* = 1.0 Tesla.

$$V=(0.1)^3=10^{-3}m^3$$
$$u_E=\frac{1}{2}\varepsilon_0E^2$$
$$u_B=\frac{B^2}{2\mu_0}$$

$$U_E=\frac{1}{2}\varepsilon_0E^2V=0.5\times8.9\times10^{-12}\times(10^5)^2\times10^{-3}$$
$$=4.5\times10^{-5}J$$

$$U_B=\frac{B^2}{2\mu_0}V=\frac{1^2}{2\times4\pi\times10^{-7}}\cdot10^{-3}$$
$$=400J$$

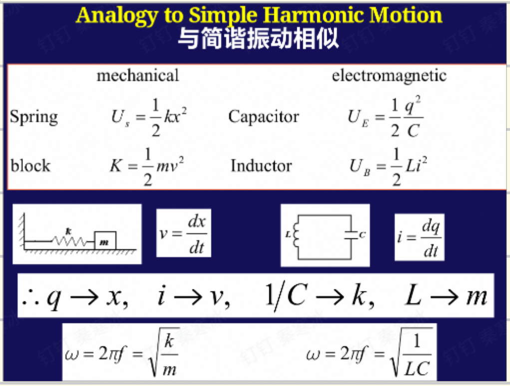
$$U_B/U_E\approx10^7$$



35-5 Electromagnetic Oscillation

（电磁振荡）

- **LC Circuits:**



Analogy to Simple Harmonic Motion

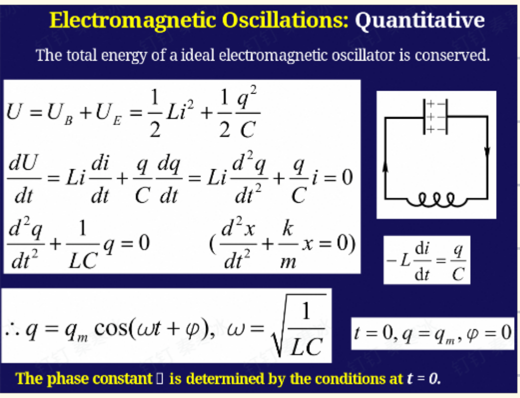
与简谐振动相似

mechanical		electromagnetic	
Spring	$U_s=\frac{1}{2}kx^2$	Capacitor	$U_E=\frac{1}{2}\frac{q^2}{C}$
block	$K=\frac{1}{2}mv^2$	Inductor	$U_B=\frac{1}{2}Li^2$

$$v=\frac{dx}{dt}\qquad i=\frac{dq}{dt}$$

$$\therefore q\to x,\quad i\to v,\quad 1/C\to k,\quad L\to m$$

$$\omega=2\pi f=\sqrt{\frac{k}{m}}\qquad \omega=2\pi f=\sqrt{\frac{1}{LC}}$$



Electromagnetic Oscillations: Quantitative

The total energy of a ideal electromagnetic oscillator is conserved.

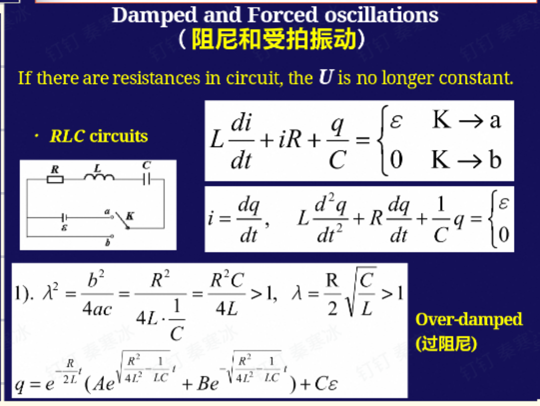
$$U=U_B+U_E=\frac{1}{2}Li^2+\frac{1}{2}\frac{q^2}{C}$$
$$\frac{dU}{dt}=Li\frac{di}{dt}+\frac{q}{C}\frac{dq}{dt}=Li\frac{d^2q}{dt^2}+\frac{q}{C}i=0$$
$$\frac{d^2q}{dt^2}+\frac{1}{LC}q=0\qquad(\frac{d^2x}{dt^2}+\frac{k}{m}x=0)$$

$$-L\frac{\mathrm{d}i}{\mathrm{d}t}=\frac{q}{C}$$

$$\therefore q=q_m\cos(\omega t+\varphi),\ \omega=\sqrt{\frac{1}{LC}}$$

$$t=0,q=q_m,\varphi=0$$

The phase constant φ is determined by the conditions at *t* = *0*.



Damped and Forced oscillations

（阻尼和受拍振动）

If there are resistances in circuit, the ***U*** is no longer constant.

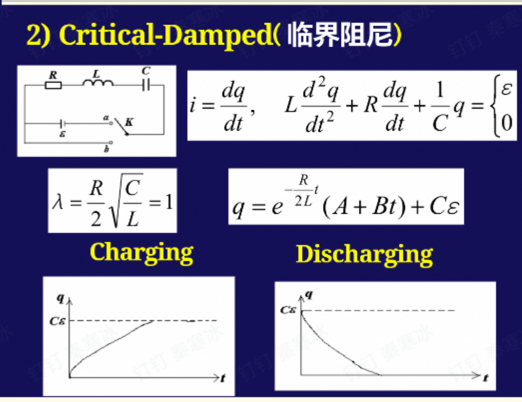
- ***RLC* circuits**

$$L\frac{di}{dt}+iR+\frac{q}{C}=\begin{cases}\varepsilon & \mathrm{K}\to\mathrm{a}\\0 & \mathrm{K}\to\mathrm{b}\end{cases}$$

$$i=\frac{dq}{dt},\quad L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=\begin{cases}\varepsilon\\0\end{cases}$$

$$1).\ \lambda^2=\frac{b^2}{4ac}=\frac{R^2}{4L\cdot\frac{1}{C}}=\frac{R^2C}{4L}>1,\ \lambda=\frac{\mathrm{R}}{2}\sqrt{\frac{C}{L}}>1$$
$$q=e^{-\frac{R}{2L}t}(Ae^{\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}t}+Be^{-\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}t})+C\varepsilon$$

Over-damped (过阻尼)



2) Critical-Damped(临界阻尼)

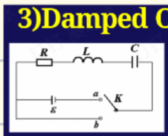
$$i=\frac{dq}{dt},\quad L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=\begin{cases}\varepsilon\\0\end{cases}$$

$$\lambda=\frac{R}{2}\sqrt{\frac{C}{L}}=1\qquad q=e^{-\frac{R}{2L}t}(A+Bt)+C\varepsilon$$

Charging

Discharging

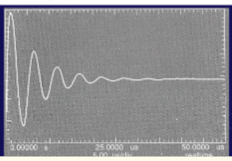
3)Damped Oscillation（阻尼振荡）



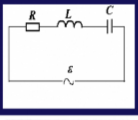
$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=\begin{cases}\varepsilon\\0\end{cases}$$

$$\lambda=\frac{R}{2}\sqrt{\frac{C}{L}}<1,\quad \beta=i\omega=i\sqrt{\frac{1}{LC}-(\frac{R}{2L})^2}=i\sqrt{\omega_0^2-(\frac{R}{2L})^2}$$

$$q=q_me^{-\frac{R}{2L}t}\cos(\omega t+\varphi)+C\varepsilon$$

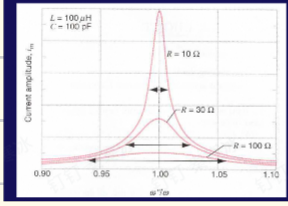


Forced Oscillations and Resonance (受拍振动和共振)



$$\text{If } \varepsilon=\varepsilon_{\mathrm{m}}\cos\omega''t$$

$$q=q_me^{-\frac{R}{2L}t}\cos(\omega t+\varphi)+C\varepsilon$$



When ω''=ω,
i_m *exhibits a maximum*

- **In vacuum:**

The Gauss' Law of Electricit y : $\oint\!\!\oint \vec{E}\bullet d\vec{A}=\frac{q_0}{\varepsilon_0}$

The Gauss' Law of Magnetism : $\oint\!\!\oint \vec{B}\bullet d\vec{A}=0$

Faraday' s Law of Induction : $\oint \vec{E}\bullet d\vec{l}=-\frac{d\Phi_B}{dt}=-\iint\frac{\partial\vec{B}}{\partial t}\bullet d\vec{A}$

Ampere' s Loop Law : $\oint \vec{B}\bullet d\vec{l}=\mu_0 i$

- **In the dielectric and magnetic materials**

$$\begin{cases}\oint\!\!\oint \vec{D}\bullet d\vec{A}=q_0\\ \oint\!\!\oint \vec{B}\bullet d\vec{A}=0\\ \oint \vec{E}\bullet d\vec{l}=-\iint\frac{\partial\vec{B}}{\partial t}\bullet d\vec{A}\\ \oint \vec{H}\bullet d\vec{l}=i_0=\iint\vec{j}\bullet d\vec{A}\end{cases}$$

Stokes 公式.

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}=\kappa_e\varepsilon_0\vec{E}$$
$$\vec{B}=\mu_0(\vec{H}+\vec{M})=\kappa_m\mu_0\vec{H}$$
$$\vec{j}_0=\sigma\vec{E}$$

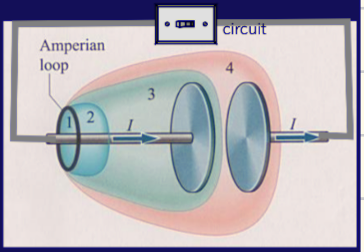
散度

自由电荷体
密度

$$\begin{cases}\nabla\bullet\vec{D}=\rho_{e0}\\ \nabla\bullet\vec{B}=0\\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\ \nabla\times\vec{H}=\vec{j}_0\end{cases}$$

旋度

Maxwell's Displacement Current

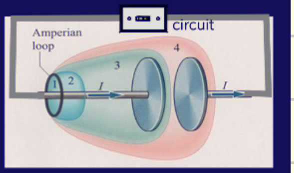


- Consider applying Ampere's Law to the current shown in the diagram.
  - If the surface is chosen as 1, 2 or 4, the enclosed current = *I*
  - If the surface is chosen as 3, the enclosed current = 0! (i.e., there is no current between the plates of the capacitor)

Big Idea: In order to have
$\oint\vec{H}\bullet d\vec{l}$ for surface 1 = $\oint\vec{H}\bullet d\vec{l}$ face 3
Maxwell proposed there was an extra "displacement current" in the region between the plates, equal to the current in the wire→

Modified Ampere's law: $\oint\vec{H}\bullet d\vec{l}=i_0+i_D$

38-2 Induced Magnetic Field (感应磁场) and the Displacement current (位移电流)



Not at steady condition

$$\oint\!\!\oint_S \vec{j}_0\bullet d\vec{A}=-\frac{dq_0}{dt}$$

From Gauss' Law:

$$\oint\!\!\oint_S \vec{D}\bullet d\vec{A}=q_0$$
$$\frac{dq_0}{dt}=\frac{d}{dt}\oint\!\!\oint_S\vec{D}\bullet d\vec{A}=\oint\!\!\oint_S\frac{\partial\vec{D}}{\partial t}\bullet d\vec{A}$$

$$\therefore\oint\!\!\oint_S\vec{j}_0\bullet d\vec{A}=-\oint\!\!\oint_S\frac{\partial\vec{D}}{\partial t}\bullet d\vec{A}$$
$$\oint\!\!\oint_S(\vec{j}_0+\frac{\partial\vec{D}}{\partial t})\bullet d\vec{A}=0$$
$$-\iint_{S_1}(\vec{j}_0+\frac{\partial\vec{D}}{\partial t})\bullet d\vec{A}=\iint_{S_2}(\vec{j}_0+\frac{\partial\vec{D}}{\partial t})\bullet d\vec{A}$$

$$\boxed{\vec{j}_0+\frac{\partial\vec{D}}{\partial t}}$$ Is continuous.

Maxwell's Displacement Current (位移电流)

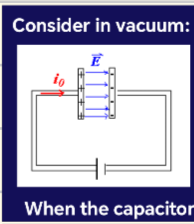
$\Phi_D=\iint\vec{D}\bullet d\vec{A}$	electric displacement flux	电位移通量
$i_D=\frac{d\Phi_D}{dt}=\iint\frac{\partial\vec{D}}{\partial t}\bullet d\vec{A}$	displacement current	位移电流
$\vec{j}_D=\frac{\partial\vec{D}}{\partial t}$	displacement current density	位移电流密度

New Ampere's Loop Law:

$$\oint\vec{H}\bullet d\vec{l}=i_0+i_D=\iint(\vec{j}_0+\frac{\partial\vec{D}}{\partial t})\bullet d\vec{A}$$

变化电场产生磁场

Consider in vacuum: In the wires, there is only conduction current i_0.
In the gap, there is only displacement current i_D.

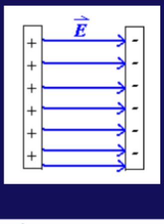


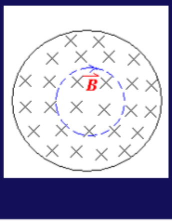
$$E=\frac{\sigma_e}{\varepsilon_0}=\frac{q}{\varepsilon_0A},\quad\therefore q=\varepsilon_0AE=\varepsilon_0\Phi_E=AD$$
$$\therefore i_0=\frac{dq}{dt}=\varepsilon_0\frac{d\Phi_E}{dt}=\frac{d\Phi_D}{dt}=i_D,\quad\vec{D}=\varepsilon_0\vec{E}$$

$\vec{D}=\varepsilon_0\vec{E}+\vec{P}$
$\vec{P}=0$

When the capacitor is fully charged, then $i_0=0$, $i_D=0$

- The induced magnetic field *B* is produced by the changing electric filed *E* inside the capacitor.

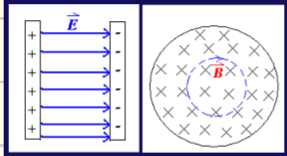




$$\oint\vec{H}\bullet d\vec{l}=\iint\frac{\partial\vec{D}}{\partial t}\bullet d\vec{A}$$
$$\oint\frac{\vec{B}}{\mu_0}\bullet d\vec{l}=\iint\varepsilon_0\frac{\partial\vec{E}}{\partial t}\bullet d\vec{A}$$
$$\oint\vec{B}\bullet d\vec{l}=\mu_0\varepsilon_0\iint\frac{\partial\vec{E}}{\partial t}\bullet d\vec{A}$$

$\frac{\partial\vec{E}}{\partial t}>0$, $\vec{B}$ is clockwise, eddy magnetic field (涡旋磁场)

Example Page863, problem 38-1



A Parallel-plate capacitor
(a) Derive an expression for the induced magnetic field at radius *r* in the region between the plates. Consider both *r* ⩽ *R* and *r* ⩾ *R*.
(b) Find *B* at *r* = *R* for *dE/dt* = 10^{12}V/m· s and *R* = 5.0 cm.

Solution:

(a). $\oint\vec{H}\bullet d\vec{l}=\iint(\vec{j}_0+\frac{\partial\vec{D}}{\partial t})\bullet d\vec{A}$

in vacuum : $\vec{j}_0=0$

$$\oint\frac{\vec{B}}{\mu_0}\bullet d\vec{l}=\iint\varepsilon_0\frac{\partial\vec{E}}{\partial t}\bullet d\vec{A}$$
$$\oint\vec{B}\bullet d\vec{l}=\mu_0\varepsilon_0\iint\frac{\partial\vec{E}}{\partial t}\bullet d\vec{A}$$

$r\leqslant R,\quad B\cdot2\pi r=\mu_0\varepsilon_0\frac{dE}{dt}\cdot\pi r^2,\quad B=\frac{1}{2}\varepsilon_0\mu_0r\frac{dE}{dt}$

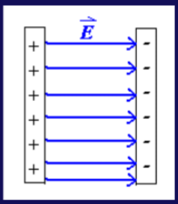
$r\geqslant R,\quad B\cdot2\pi r=\mu_0\varepsilon_0\frac{dE}{dt}\cdot\pi R^2,\quad B=\frac{1}{2}\varepsilon_0\mu_0\frac{R^2}{r}\frac{dE}{dt}$

(b) $r=R,\ \frac{dE}{dt}=10^{12}V/m\cdot s$

$$B=\frac{1}{2}\varepsilon_0\mu_0R\frac{dE}{dt}$$
$$=\frac{1}{2}(8.9\cdot10^{-12}C^2/Nm^2)\cdot(4\pi\cdot10^{-7}T.m/A)\cdot(5.0\cdot10^{-2})(10^{12})$$
$$=2.8\cdot10^{-7}T$$
$$=280nT$$

They can scarcely be measured with simple apparatus.

- What is the displacement current for the situation of Sample Problem 38-1?

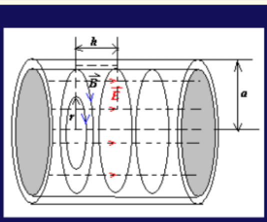


Solution:

$$i_D = \frac{d\Phi_D}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \pi R^2 \cdot \frac{dE}{dt}$$
$$= 8.9 \cdot 10^{-12} \cdot \pi \cdot (5 \cdot 10^{-2})^2 \cdot 10^{12}$$
$$= 0.07\text{A} \ = 70\text{mA}$$

i_D is a reasonably large current, but B=280 nT, Why?

Under the same conditions, both kinds of current are equally effective in generating magnetic field.



A more detailed representation of a cylindrical electromagnetic resonant cavity

- From Faraday's Law

$$\oint \vec{E} \bullet d\vec{l} = -\frac{d\Phi_B}{dt}$$
$$B \text{ increasing},\ E \cdot h = -\frac{d\Phi_B}{dt}$$
$$E = -\frac{1}{h}\frac{d\Phi_B}{dt}$$

- From Ampere's Loop Law

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 \varepsilon_0 \iint \frac{\partial \vec{E}}{\partial t} \bullet d\vec{A}$$
$$B \cdot 2\pi r = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$
$$B = \frac{\mu_0 \varepsilon_0}{2\pi r}\frac{d\Phi_E}{dt}$$

38-3 Maxwell's Equations

- In vacuum:

The Gauss' Law of Electricit y : $\oiint \vec{E} \bullet d\vec{A} = \frac{q_0}{\varepsilon_0}$

The Gauss' Law of Magnetism : $\oiint \vec{B} \bullet d\vec{A} = 0$

Faraday' s Law of Induction : $\oint \vec{E} \bullet d\vec{l} = -\frac{d\Phi_B}{dt} = -\iint \frac{\partial \vec{B}}{\partial t} \bullet d\vec{A}$

Ampere' s Loop Law : $\oint \vec{B} \bullet d\vec{l} = \mu_0 i + \mu_0 \varepsilon_0 \iint \frac{\partial \vec{E}}{\partial t} \bullet d\vec{A}$

(As extended by Maxwell)

- In the dielectric and magnetic materials

$$\begin{cases} \oiint \vec{D} \bullet d\vec{A} = q_0 \\ \oiint \vec{B} \bullet d\vec{A} = 0 \\ \oint \vec{E} \bullet d\vec{l} = -\iint \frac{\partial \vec{B}}{\partial t} \bullet d\vec{A} \\ \oint \vec{H} \bullet d\vec{l} = i_0 + \iint \frac{\partial \vec{D}}{\partial t} \bullet d\vec{A} \end{cases}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \kappa_e \varepsilon_0 \vec{E}$$
$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \kappa_m \mu_0 \vec{H}$$
$$\vec{j}_0 = \sigma \vec{E}$$

$$\begin{cases} \nabla \bullet \vec{D} = \rho_{e0} \\ \nabla \bullet \vec{B} = 0 \\ \nabla \cdot \ \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \ \vec{H} = \vec{j}_0 + \frac{\partial \vec{D}}{\partial t} \end{cases}$$

电磁波

2. The emitting of Electromagnetic Wave 电磁波的发射

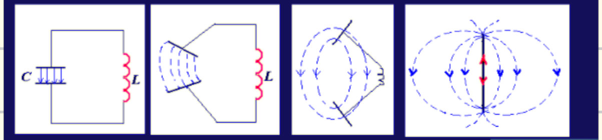
The condition of emitting electromagnetic wave:

(1) The frequency of electromagnetic wave has to be very high:

$$\frac{dW}{dt} \propto f^4 \qquad f > 10^5\ Hz$$
$$\because\ f_0 = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$
$$\therefore\ L, C \ \text{ have to be very small}$$

(2) The LC circuit must be opened:

L, C are the distributed element $f_0 = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$



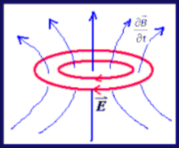
L, *C* is decreasing, the circuit is opening.

A wire, *i* surge back and forth in the wire.

Dipole antenna (偶极振子天线)

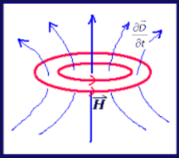
3. The transmission of EMW (电磁波的传播)

- Transmission Medium? Ether, Aether (以太)
- It is not necessary to have medium for the transmission of electromagnetic wave.



Changing a magnetic field produce and an electric field

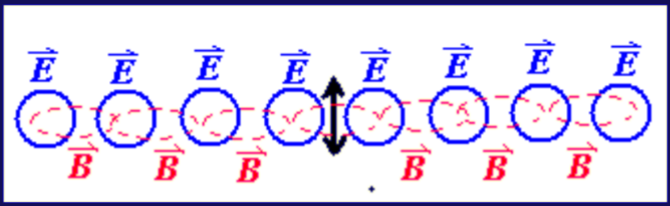
$$\oint \vec{E} \bullet d\vec{l} = -\iint \frac{\partial \vec{B}}{\partial t} \bullet d\vec{A}$$



Changing an electric field produce and a magnetic field

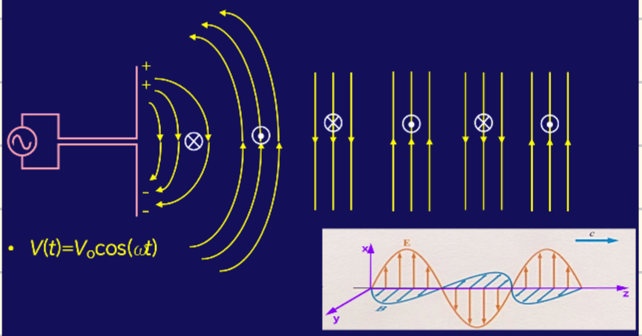
$$\oint \vec{H} \bullet d\vec{l} = \iint \frac{\partial \vec{D}}{\partial t} \bullet d\vec{A}$$

Vortex electric field, Vortex magnetic field (涡旋电场和涡旋磁场)



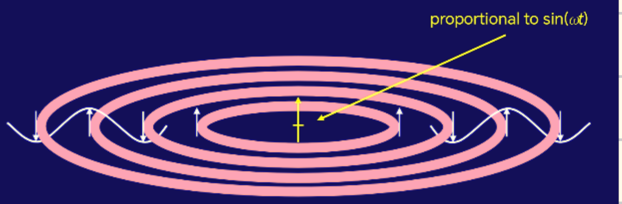
Electromagnetic Wave

Radiation from oscillating dipole



- $V(t)=V_0\cos(\omega t)$
- Wave propagates outward at speed of light

Dipole radiation pattern



- Oscillating electric dipole (振荡的电偶极矩) generates e-m radiation that is **linearly polarized (线偏振)** in the direction of the dipole.
- Radiation pattern is doughnut shaped & outward traveling
  - zero amplitude above and below dipole.
  - maximum amplitude in-plane.

39-4 The properties of electromagnetic wave

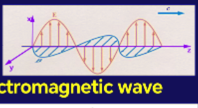
At distant from the wave source, there are *5* properties

In free space (自由空间): $\rho_{e0}=0,\ \vec{j}_0=0$

1. Horizontal Wave (横波)

$\vec{E}\perp\vec{k},\ \ \vec{H}\perp\vec{k}$

2. $\vec{E}\perp\vec{H}$
3. *E*, *H* are in phase (同相)
4. Right-Hand rule



$\sqrt{\kappa_e\varepsilon_0}\,E_0=\sqrt{\kappa_m\mu_0}\,H_0$

5. The speed of electromagnetic wave

$v=\dfrac{1}{\sqrt{\kappa_e\varepsilon_0\kappa_m\mu_0}}$

in the air (vacuum), $\kappa_e=\kappa_m=1,\ \ v=c=\dfrac{1}{\sqrt{\varepsilon_0\mu_0}}=3\cdot 10^8\ m/s$

$$\begin{cases}\oiint\vec{D}\cdot d\vec{A}=q_0 & \nabla\cdot\vec{D}=\rho_{e0}\\ \oiint\vec{B}\cdot d\vec{A}=0 & \nabla\cdot\vec{B}=0\\ \oint\vec{E}\cdot d\vec{l}=-\iint\frac{\partial\vec{B}}{\partial t}\cdot d\vec{A} & \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\ \oint\vec{H}\cdot d\vec{l}=i_0+\iint\frac{\partial\vec{D}}{\partial t}\cdot d\vec{A} & \nabla\times\vec{H}=\vec{j}_{e0}+\frac{\partial\vec{D}}{\partial t}\end{cases}$$

自由电荷体密度　真空中：

$$\nabla\cdot\vec{D}=\rho_{e0}\xrightarrow{\rho_{e0}=0}\nabla\cdot\vec{E}=0$$

$$\nabla\cdot\vec{B}=0\longrightarrow\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\xrightarrow{\vec{B}=k_m\mu_0\vec{H}}\nabla\times\vec{E}=-k_m\mu_0\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H}=\vec{j}_0+\frac{\partial\vec{D}}{\partial t}\xrightarrow{j_0=0}\nabla\times\vec{H}=k_e\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

统一成 $\vec{E}$ 与 $\vec{H}$　$\vec{B}=k_m\mu_0\vec{H}$　$\vec{D}=k_e\varepsilon_0\vec{E}$

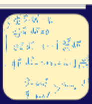
$$\nabla\cdot\vec{E}=0\qquad \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0$$

$$\Rightarrow \underline{\nabla\cdot\vec{B}=0}\ \ \nabla\cdot\vec{H}=0\qquad \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$$

$$\nabla\times\vec{E}=-k_m\mu_0\frac{\partial\vec{H}}{\partial t}\qquad \begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ E_x&E_y&E_z\end{vmatrix}=-k_m\mu_0\left(\frac{\partial H_x}{\partial t}\vec{i}+\frac{\partial H_y}{\partial t}\vec{j}+\frac{\partial H_z}{\partial t}\vec{k}\right).$$

$$\nabla\times\vec{H}=k_e\varepsilon_0\frac{\partial\vec{E}}{\partial t}\qquad \begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ H_x&H_y&H_z\end{vmatrix}=K_e\varepsilon_0\left(\frac{\partial E_x}{\partial t}\vec{i}+\frac{\partial E_y}{\partial t}\vec{j}+\frac{\partial E_z}{\partial t}\vec{k}\right).$$

For the plane wave (平面波)

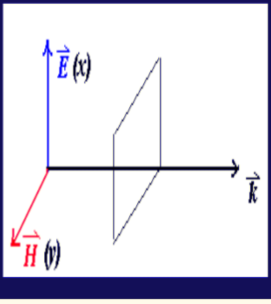


$$\begin{cases}\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 & (1)\\ \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\kappa_m\mu_0\frac{\partial H_x}{\partial t} & (2\text{-}1)\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t} & (2\text{-}2)\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\kappa_m\mu_0\frac{\partial H_z}{\partial t} & (2\text{-}3)\\ \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0 & (3)\\ \frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=\kappa_e\varepsilon_0\frac{\partial E_x}{\partial t} & (4\text{-}1)\\ \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=\kappa_e\varepsilon_0\frac{\partial E_y}{\partial t} & (4\text{-}2)\\ \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=\kappa_e\varepsilon_0\frac{\partial E_z}{\partial t} & (4\text{-}3)\end{cases}$$

$\vec{k}$, + z axis direction

The phase in the wave plane is the same, φ is independent of *x*, *y*.

For simplicity, $\vec{E}$, $\vec{H}$ is independent of *x*, *y*.



(1) Horizontal wave（横波）

$$\begin{cases}\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 & (1)\\ \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\kappa_m\mu_0\frac{\partial H_x}{\partial t} & (2\text{-}1)\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t} & (2\text{-}2)\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\kappa_m\mu_0\frac{\partial H_z}{\partial t} & (2\text{-}3)\\ \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0 & (3)\\ \frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=\kappa_e\varepsilon_0\frac{\partial E_x}{\partial t} & (4\text{-}1)\\ \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=\kappa_e\varepsilon_0\frac{\partial E_y}{\partial t} & (4\text{-}2)\\ \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=\kappa_e\varepsilon_0\frac{\partial E_z}{\partial t} & (4\text{-}3)\end{cases}$$

$$\text{From }(1)\Rightarrow\frac{\partial E_z}{\partial z}=0$$

$$(2-3)\Rightarrow\frac{\partial H_z}{\partial t}=0$$

$$(3)\Rightarrow\frac{\partial H_z}{\partial z}=0$$

$$(4-3)\Rightarrow\frac{\partial E_z}{\partial t}=0$$

$$\longrightarrow\quad H_z(z,t)=\text{constant}\qquad E_z(z,t)=\text{constant}$$

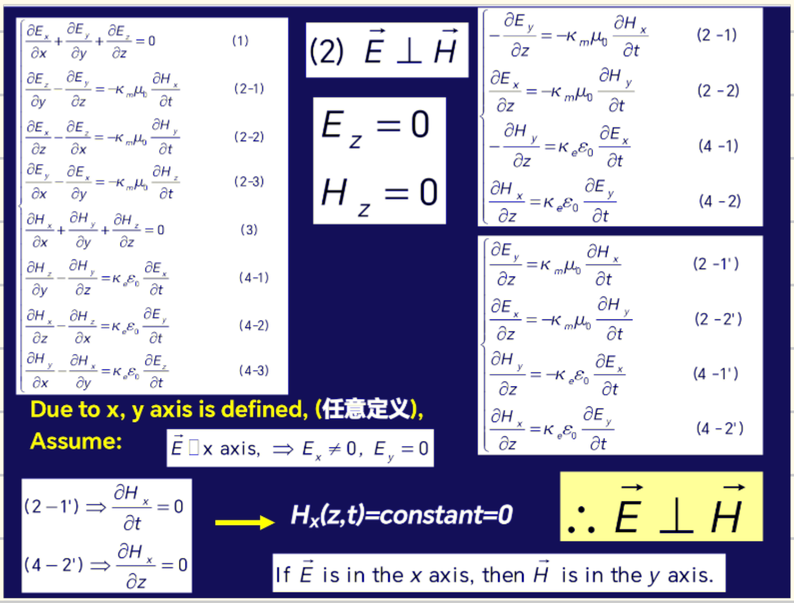
E$_z$, *H*$_z$ is is independent of *EMW*.
So, assume $E_z=0,\ H_z=0$

The electromagnetic wave is horizontal wave.

$\vec{E}\perp\vec{k},\ \ \vec{H}\perp\vec{k}$

因为平面波沿+z方向传播　∴ $\vec{E}$与$\vec{H}$分布与x, y无关，仅随z和t变化 $\frac{\partial}{\partial x}=0$，$\frac{\partial}{\partial y}=0$ ⇒ $\frac{\partial H_z}{\partial t}=\frac{\partial H_z}{\partial z}=\frac{\partial E_z}{\partial z}=\frac{\partial E_z}{\partial t}=0$

⇒ $H_z(z,t)$=常数，$E_z(z,t)$=常数　⇒ $\vec{E}=\vec{E}_x+\vec{E}_y$　$\vec{H}=\vec{H}_x+\vec{H}_y$　⇒ $\vec{E}\perp\vec{z}$　$\vec{H}\perp\vec{z}$ (即垂直于$\vec{k}$)



$$\begin{cases}\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 & (1)\\ \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\kappa_m\mu_0\frac{\partial H_x}{\partial t} & (2\text{-}1)\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t} & (2\text{-}2)\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\kappa_m\mu_0\frac{\partial H_z}{\partial t} & (2\text{-}3)\\ \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0 & (3)\\ \frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=\kappa_e\varepsilon_0\frac{\partial E_x}{\partial t} & (4\text{-}1)\\ \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=\kappa_e\varepsilon_0\frac{\partial E_y}{\partial t} & (4\text{-}2)\\ \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=\kappa_e\varepsilon_0\frac{\partial E_z}{\partial t} & (4\text{-}3)\end{cases}$$

(2) $\vec{E}\perp\vec{H}$

$$E_z=0$$
$$H_z=0$$

$$\begin{cases}-\frac{\partial E_y}{\partial z}=-\kappa_m\mu_0\frac{\partial H_x}{\partial t} & (2\text{-}1)\\ \frac{\partial E_x}{\partial z}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t} & (2\text{-}2)\\ -\frac{\partial H_y}{\partial z}=\kappa_e\varepsilon_0\frac{\partial E_x}{\partial t} & (4\text{-}1)\\ \frac{\partial H_x}{\partial z}=\kappa_e\varepsilon_0\frac{\partial E_y}{\partial t} & (4\text{-}2)\end{cases}$$

$$\begin{cases}\frac{\partial E_y}{\partial z}=\kappa_m\mu_0\frac{\partial H_x}{\partial t} & (2\text{-}1')\\ \frac{\partial E_x}{\partial z}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t} & (2\text{-}2')\\ \frac{\partial H_y}{\partial z}=-\kappa_e\varepsilon_0\frac{\partial E_x}{\partial t} & (4\text{-}1')\\ \frac{\partial H_x}{\partial z}=\kappa_e\varepsilon_0\frac{\partial E_y}{\partial t} & (4\text{-}2')\end{cases}$$

Due to x, y axis is defined, (任意定义),

Assume: $\vec{E}$ □x axis, $\Rightarrow E_x\neq0,\ E_y=0$

$$(2-1')\Rightarrow\frac{\partial H_x}{\partial t}=0$$
$$(4-2')\Rightarrow\frac{\partial H_x}{\partial z}=0$$

→ ***$H_x(z,t)$=constant=0***

$$\therefore\vec{E}\perp\vec{H}$$

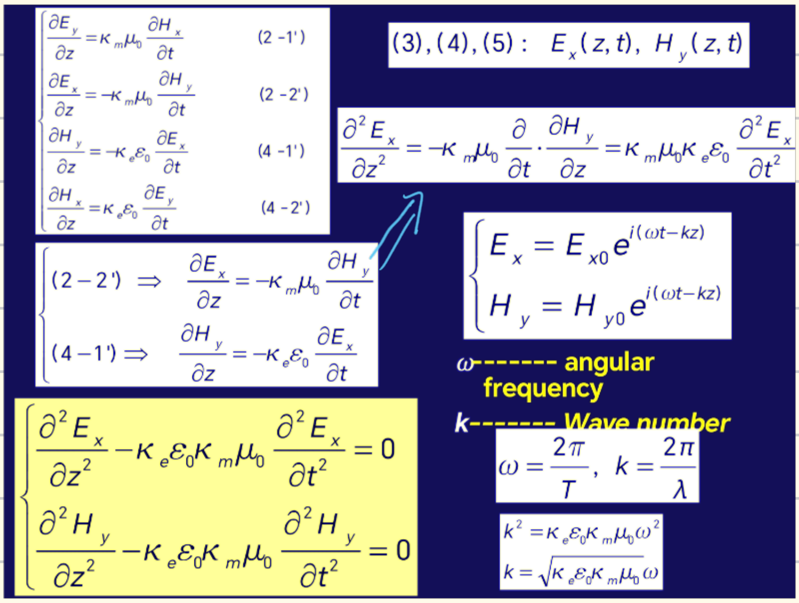
If $\vec{E}$ is in the x axis, then $\vec{H}$ is in the y axis.

假设 $E_x\neq0$，$E_y=0$

⇒ $\frac{\partial H_x}{\partial z}=0$，$\frac{\partial H_x}{\partial t}=0$　$H_x(z,t)$=常数 . ⇒ $H_y\neq0$

assume=0

$$\left.\begin{aligned}&\Rightarrow\vec{E}=E_x\vec{i}\\&\vec{H}=H_y\vec{j}\end{aligned}\right\rangle\ \vec{E}\perp\vec{H}$$



$$\begin{cases}\frac{\partial E_y}{\partial z}=\kappa_m\mu_0\frac{\partial H_x}{\partial t} & (2\text{-}1')\\ \frac{\partial E_x}{\partial z}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t} & (2\text{-}2')\\ \frac{\partial H_y}{\partial z}=-\kappa_e\varepsilon_0\frac{\partial E_x}{\partial t} & (4\text{-}1')\\ \frac{\partial H_x}{\partial z}=\kappa_e\varepsilon_0\frac{\partial E_y}{\partial t} & (4\text{-}2')\end{cases}$$

(3), (4), (5) :　$E_x(z,t),\ H_y(z,t)$

$$\begin{cases}(2-2')\Rightarrow\frac{\partial E_x}{\partial z}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t}\\ (4-1')\Rightarrow\frac{\partial H_y}{\partial z}=-\kappa_e\varepsilon_0\frac{\partial E_x}{\partial t}\end{cases}$$

$$\frac{\partial^2E_x}{\partial z^2}=-\kappa_m\mu_0\frac{\partial}{\partial t}\cdot\frac{\partial H_y}{\partial z}=\kappa_m\mu_0\kappa_e\varepsilon_0\frac{\partial^2E_x}{\partial t^2}$$

$$\begin{cases}\frac{\partial^2E_x}{\partial z^2}-\kappa_e\varepsilon_0\kappa_m\mu_0\frac{\partial^2E_x}{\partial t^2}=0\\ \frac{\partial^2H_y}{\partial z^2}-\kappa_e\varepsilon_0\kappa_m\mu_0\frac{\partial^2H_y}{\partial t^2}=0\end{cases}$$

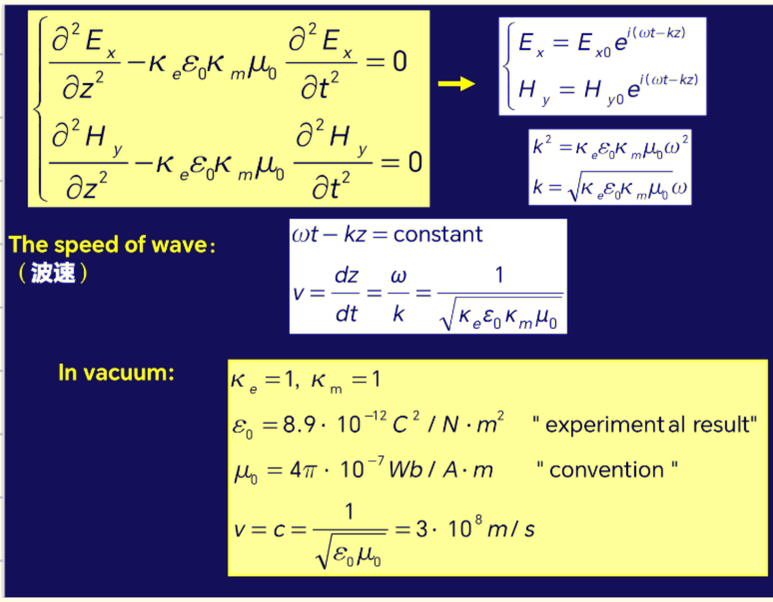
$$\begin{cases}E_x=E_{x0}e^{i(\omega t-kz)}\\ H_y=H_{y0}e^{i(\omega t-kz)}\end{cases}$$

ω------- angular frequency

k------- Wave number

$$\omega=\frac{2\pi}{T},\quad k=\frac{2\pi}{\lambda}$$

$$k^2=\kappa_e\varepsilon_0\kappa_m\mu_0\omega^2$$
$$k=\sqrt{\kappa_e\varepsilon_0\kappa_m\mu_0}\,\omega$$



$$\begin{cases}\frac{\partial^2E_x}{\partial z^2}-\kappa_e\varepsilon_0\kappa_m\mu_0\frac{\partial^2E_x}{\partial t^2}=0\\ \frac{\partial^2H_y}{\partial z^2}-\kappa_e\varepsilon_0\kappa_m\mu_0\frac{\partial^2H_y}{\partial t^2}=0\end{cases}\rightarrow\begin{cases}E_x=E_{x0}e^{i(\omega t-kz)}\\ H_y=H_{y0}e^{i(\omega t-kz)}\end{cases}$$

$$k^2=\kappa_e\varepsilon_0\kappa_m\mu_0\omega^2$$
$$k=\sqrt{\kappa_e\varepsilon_0\kappa_m\mu_0}\,\omega$$

The speed of wave:
（波速）

$\omega t-kz=$ constant

$$v=\frac{dz}{dt}=\frac{\omega}{k}=\frac{1}{\sqrt{\kappa_e\varepsilon_0\kappa_m\mu_0}}$$

In vacuum:

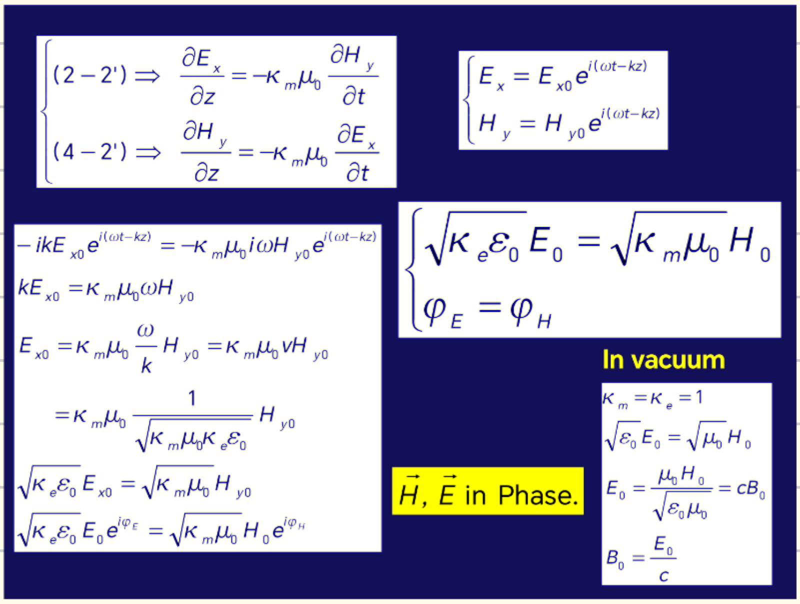
$\kappa_e=1,\ \kappa_m=1$

$\varepsilon_0=8.9\cdot10^{-12}C^2/N\cdot m^2$　" experimental result"

$\mu_0=4\pi\cdot10^{-7}Wb/A\cdot m$　" convention "

$$v=c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}=3\cdot10^8m/s$$

一維波动方程：$\frac{\partial^2y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2y}{\partial t^2}$　⇒ $\frac{1}{v^2}=k_e\varepsilon_0k_m\mu_0$



$$\begin{cases}(2-2')\Rightarrow\frac{\partial E_x}{\partial z}=-\kappa_m\mu_0\frac{\partial H_y}{\partial t}\\ (4-2')\Rightarrow\frac{\partial H_y}{\partial z}=-\kappa_m\mu_0\frac{\partial E_x}{\partial t}\end{cases}$$

$$\begin{cases}E_x=E_{x0}e^{i(\omega t-kz)}\\ H_y=H_{y0}e^{i(\omega t-kz)}\end{cases}$$

$$-ikE_{x0}e^{i(\omega t-kz)}=-\kappa_m\mu_0i\omega H_{y0}e^{i(\omega t-kz)}$$
$$kE_{x0}=\kappa_m\mu_0\omega H_{y0}$$
$$E_{x0}=\kappa_m\mu_0\frac{\omega}{k}H_{y0}=\kappa_m\mu_0vH_{y0}$$
$$=\kappa_m\mu_0\frac{1}{\sqrt{\kappa_m\mu_0\kappa_e\varepsilon_0}}H_{y0}$$
$$\sqrt{\kappa_e\varepsilon_0}E_{x0}=\sqrt{\kappa_m\mu_0}H_{y0}$$
$$\sqrt{\kappa_e\varepsilon_0}E_0e^{i\varphi_E}=\sqrt{\kappa_m\mu_0}H_0e^{i\varphi_H}$$

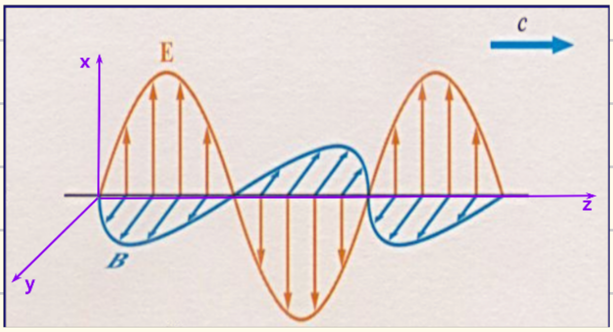
$$\begin{cases}\sqrt{\kappa_e\varepsilon_0}E_0=\sqrt{\kappa_m\mu_0}H_0\\ \varphi_E=\varphi_H\end{cases}$$

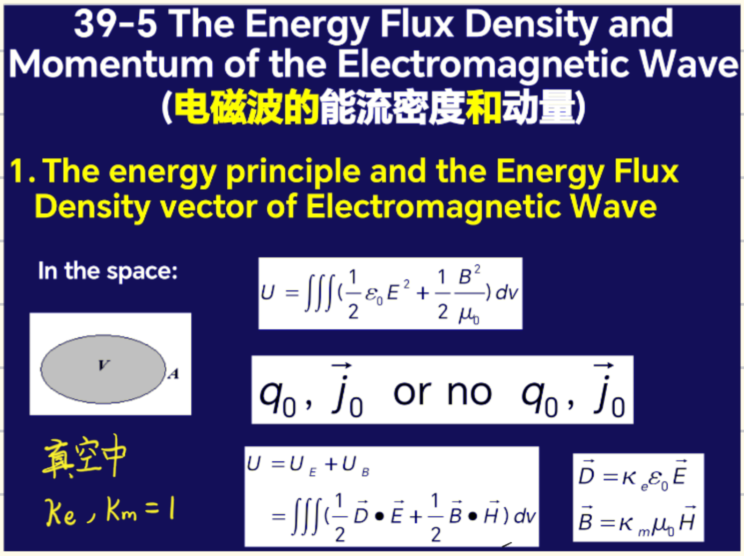
$\vec{H}$, $\vec{E}$ in Phase.

In vacuum

$\kappa_m=\kappa_e=1$

$$\sqrt{\varepsilon_0}E_0=\sqrt{\mu_0}H_0$$
$$E_0=\frac{\mu_0H_0}{\sqrt{\varepsilon_0\mu_0}}=cB_0$$
$$B_0=\frac{E_0}{c}$$





39-5 The Energy Flux Density and Momentum of the Electromagnetic Wave (电磁波的能流密度和动量)

1. The energy principle and the Energy Flux Density vector of Electromagnetic Wave

In the space:

$$U=\iiint\left(\frac{1}{2}\varepsilon_0E^2+\frac{1}{2}\frac{B^2}{\mu_0}\right)dv$$

$q_0,\ \vec{j}_0$ or no $q_0,\ \vec{j}_0$

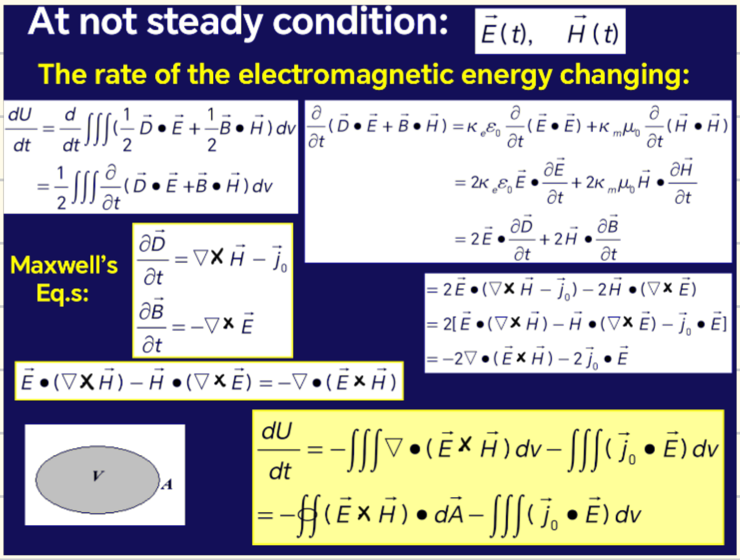
$$U=U_E+U_B=\iiint\left(\frac{1}{2}\vec{D}\bullet\vec{E}+\frac{1}{2}\vec{B}\bullet\vec{H}\right)dv$$

$\vec{D}=\kappa_e\varepsilon_0\vec{E}$

$\vec{B}=\kappa_m\mu_0\vec{H}$

真空中

$\kappa_e,\ \kappa_m=1$



At not steady condition: $\vec{E}(t),\quad \vec{H}(t)$

The rate of the electromagnetic energy changing:

$$\frac{dU}{dt}=\frac{d}{dt}\iiint\left(\frac{1}{2}\vec{D}\bullet\vec{E}+\frac{1}{2}\vec{B}\bullet\vec{H}\right)dv=\frac{1}{2}\iiint\frac{\partial}{\partial t}(\vec{D}\bullet\vec{E}+\vec{B}\bullet\vec{H})dv$$

$$\frac{\partial}{\partial t}(\vec{D}\bullet\vec{E}+\vec{B}\bullet\vec{H})=\kappa_e\varepsilon_0\frac{\partial}{\partial t}(\vec{E}\bullet\vec{E})+\kappa_m\mu_0\frac{\partial}{\partial t}(\vec{H}\bullet\vec{H})=2\kappa_e\varepsilon_0\vec{E}\bullet\frac{\partial\vec{E}}{\partial t}+2\kappa_m\mu_0\vec{H}\bullet\frac{\partial\vec{H}}{\partial t}=2\vec{E}\bullet\frac{\partial\vec{D}}{\partial t}+2\vec{H}\bullet\frac{\partial\vec{B}}{\partial t}$$

Maxwell's Eq.s:

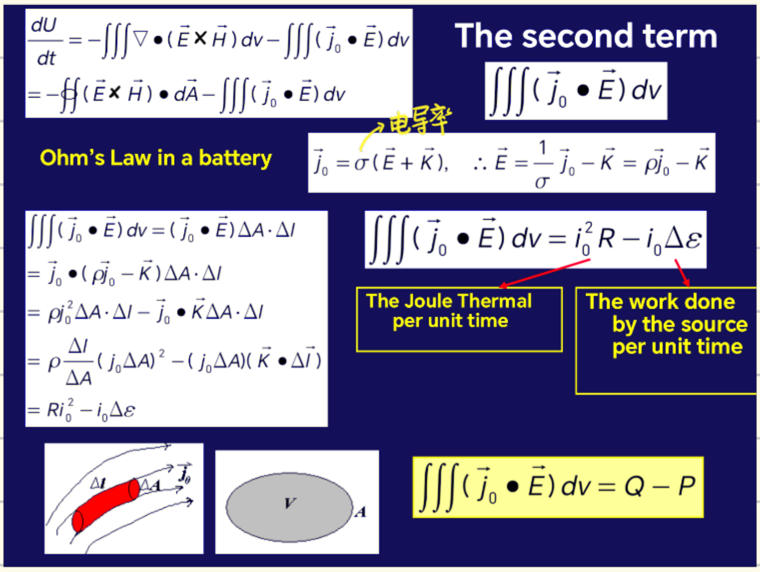
$$\frac{\partial\vec{D}}{\partial t}=\nabla\times\vec{H}-\vec{j}_0$$

$$\frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E}$$

$$=2\vec{E}\bullet(\nabla\times\vec{H}-\vec{j}_0)-2\vec{H}\bullet(\nabla\times\vec{E})=2[\vec{E}\bullet(\nabla\times\vec{H})-\vec{H}\bullet(\nabla\times\vec{E})-\vec{j}_0\bullet\vec{E}]=-2\nabla\bullet(\vec{E}\times\vec{H})-2\vec{j}_0\bullet\vec{E}$$

$$\vec{E}\bullet(\nabla\times\vec{H})-\vec{H}\bullet(\nabla\times\vec{E})=-\nabla\bullet(\vec{E}\times\vec{H})$$

$$\frac{dU}{dt}=-\iiint\nabla\bullet(\vec{E}\times\vec{H})dv-\iiint(\vec{j}_0\bullet\vec{E})dv=-\oint\!\!\oint(\vec{E}\times\vec{H})\bullet d\vec{A}-\iiint(\vec{j}_0\bullet\vec{E})dv$$



$$\frac{dU}{dt}=-\iiint\nabla\bullet(\vec{E}\times\vec{H})dv-\iiint(\vec{j}_0\bullet\vec{E})dv=-\oint\!\!\oint(\vec{E}\times\vec{H})\bullet d\vec{A}-\iiint(\vec{j}_0\bullet\vec{E})dv$$

The second term

$$\iiint(\vec{j}_0\bullet\vec{E})dv$$

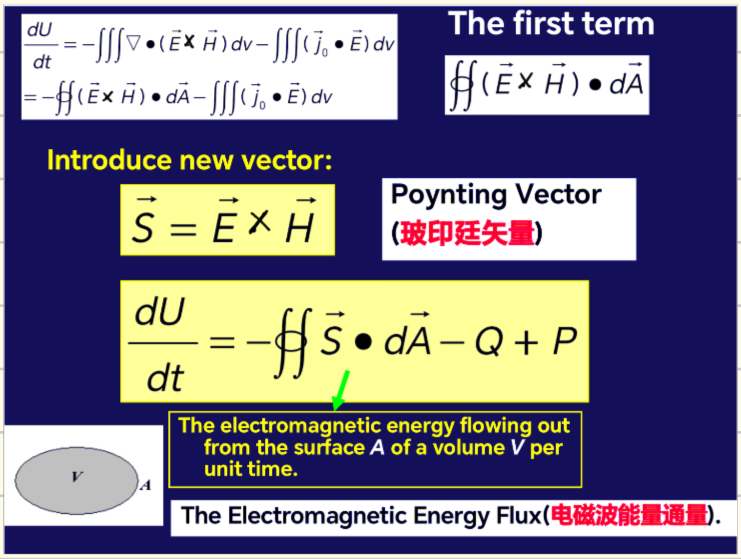
Ohm's Law in a battery

$$\vec{j}_0=\overset{\text{电导率}}{\sigma}(\vec{E}+\vec{K}),\quad \therefore\vec{E}=\frac{1}{\sigma}\vec{j}_0-\vec{K}=\rho\vec{j}_0-\vec{K}$$

$$\iiint(\vec{j}_0\bullet\vec{E})dv=(\vec{j}_0\bullet\vec{E})\Delta A\cdot\Delta l=\vec{j}_0\bullet(\rho\vec{j}_0-\vec{K})\Delta A\cdot\Delta l=\rho j_0^2\Delta A\cdot\Delta l-\vec{j}_0\bullet\vec{K}\Delta A\cdot\Delta l=\rho\frac{\Delta l}{\Delta A}(j_0\Delta A)^2-(j_0\Delta A)(\vec{K}\bullet\Delta\vec{l})=Ri_0^2-i_0\Delta\varepsilon$$

$$\iiint(\vec{j}_0\bullet\vec{E})dv=\underbrace{i_0^2R}_{\text{The Joule Thermal per unit time}}-\underbrace{i_0\Delta\varepsilon}_{\text{The work done by the source per unit time}}$$

$$\iiint(\vec{j}_0\bullet\vec{E})dv=Q-P$$



$$\frac{dU}{dt}=-\iiint\nabla\bullet(\vec{E}\times\vec{H})dv-\iiint(\vec{j}_0\bullet\vec{E})dv=-\oint\!\!\oint(\vec{E}\times\vec{H})\bullet d\vec{A}-\iiint(\vec{j}_0\bullet\vec{E})dv$$

The first term

$$\oint\!\!\oint(\vec{E}\times\vec{H})\bullet d\vec{A}$$

Introduce new vector:

$$\vec{S}=\vec{E}\times\vec{H}$$

Poynting Vector (玻印廷矢量)

$$\frac{dU}{dt}=-\oint\!\!\oint\vec{S}\bullet d\vec{A}-Q+P$$

The electromagnetic energy flowing out from the surface *A* of a volume *V* per unit time.

The Electromagnetic Energy Flux(电磁波能量通量).

P: 电源做功功率　　Q: 焦耳热功率

$\vec{S}=\vec{E}\times\vec{H}$ 玻印廷矢量：单位时间内通过单位面积的电磁能量.

$u_E=\frac{1}{2}\varepsilon_0E^2$　$u_B=\frac{B^2}{2\mu_0}$;　$\sqrt{\kappa_m\mu_0}H=\sqrt{\kappa_e\varepsilon_0}E$

$$\Downarrow$$

$$\sqrt{\mu_0}H=\sqrt{\varepsilon_0}E$$

$\therefore u_E=\frac{1}{2}\varepsilon_0E^2=\frac{1}{2}\varepsilon_0\cdot c^2B^2=\frac{1}{2}\varepsilon_0\cdot\frac{1}{\varepsilon_0\mu_0}B^2=\frac{B^2}{2\mu_0}=u_B$　∴ 电磁波中电场能量密度与磁场能量密度相等.

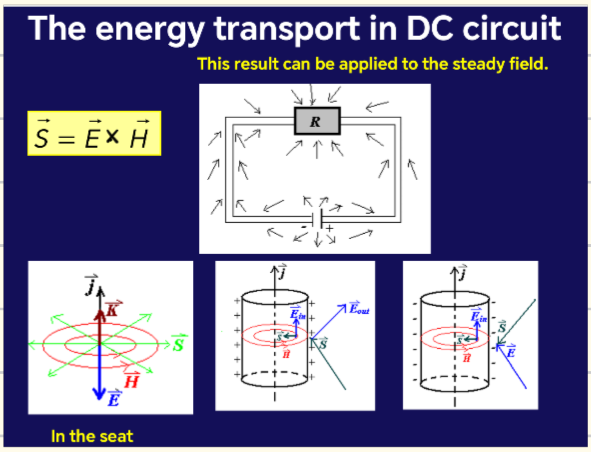
$u=u_B+u_E=\varepsilon_0E^2$

$B_0=\frac{E_0}{c}$

⇒平均能量密度: $\langle u\rangle=\varepsilon_0\langle E^2\rangle=\varepsilon_0E_{max}^2\langle\sin^2(wt-kx)\rangle=\frac{\varepsilon_0E_{max}^2}{2}$　　$E_{rms}=\frac{E_{max}}{\sqrt{2}}$　　$\langle u\rangle=\varepsilon_0E_{rms}^2$

波的强度 $I=c\langle u\rangle=c\cdot\frac{\varepsilon_0E_{max}^2}{2}$　I: 单位时间内通过单位面积的电磁能

$$\left.\begin{cases}I=c\langle u\rangle\\ \langle u\rangle=\varepsilon_0\langle E^2\rangle=\frac{\langle B^2\rangle}{\mu_0}\Rightarrow\varepsilon_0E_{rms}^2=\frac{B_{rms}^2}{\mu_0}\quad B_{rms}=\frac{E_{rms}}{c}\end{cases}\right\rangle\ I=C\cdot\varepsilon_0\cdot E_{rms}^2=\frac{1}{\mu_0C}E_{rms}^2$$

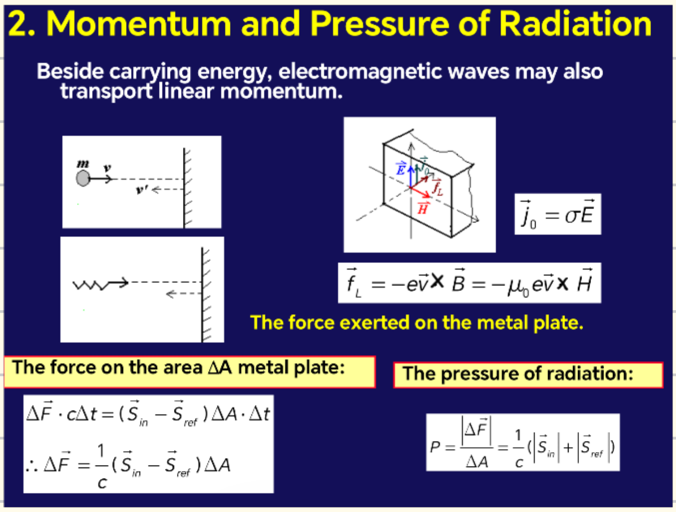


The energy transport in DC circuit

This result can be applied to the steady field.

$\vec{S}=\vec{E}\times\vec{H}$

In the seat



2. Momentum and Pressure of Radiation

Beside carrying energy, electromagnetic waves may also transport linear momentum.

$$\vec{j}_0 = \sigma\vec{E}$$

$$\vec{f}_L = -e\vec{v} \times \vec{B} = -\mu_0 e\vec{v} \times \vec{H}$$

The force exerted on the metal plate.

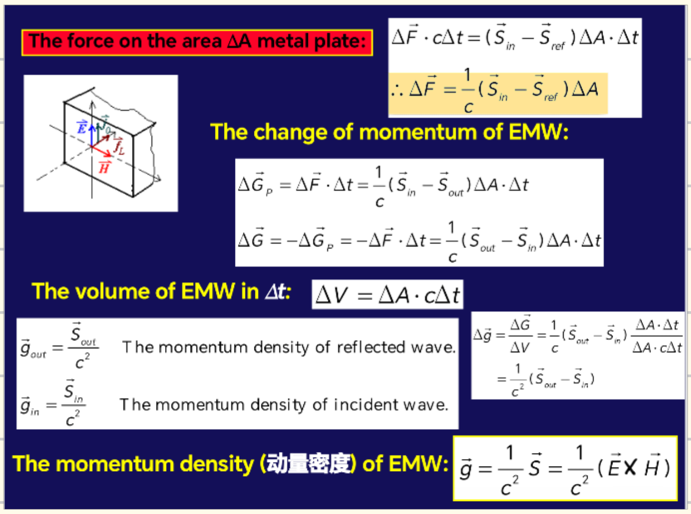
The force on the area ΔA metal plate:

$$\Delta\vec{F}\cdot c\Delta t = (\vec{S}_{in} - \vec{S}_{ref})\Delta A\cdot\Delta t$$

$$\therefore \Delta\vec{F} = \frac{1}{c}(\vec{S}_{in} - \vec{S}_{ref})\Delta A$$

The pressure of radiation:

$$P = \frac{|\Delta\vec{F}|}{\Delta A} = \frac{1}{c}(|\vec{S}_{in}| + |\vec{S}_{ref}|)$$



The force on the area ΔA metal plate:

$$\Delta\vec{F}\cdot c\Delta t = (\vec{S}_{in} - \vec{S}_{ref})\Delta A\cdot\Delta t$$

$$\therefore \Delta\vec{F} = \frac{1}{c}(\vec{S}_{in} - \vec{S}_{ref})\Delta A$$

The change of momentum of EMW:

$$\Delta\vec{G}_P = \Delta\vec{F}\cdot\Delta t = \frac{1}{c}(\vec{S}_{in} - \vec{S}_{out})\Delta A\cdot\Delta t$$

$$\Delta\vec{G} = -\Delta\vec{G}_P = -\Delta\vec{F}\cdot\Delta t = \frac{1}{c}(\vec{S}_{out} - \vec{S}_{in})\Delta A\cdot\Delta t$$

The volume of EMW in *Δt*: $\Delta V = \Delta A\cdot c\Delta t$

$\vec{g}_{out} = \frac{\vec{S}_{out}}{c^2}$ The momentum density of reflected wave.

$\vec{g}_{in} = \frac{\vec{S}_{in}}{c^2}$ The momentum density of incident wave.

$$\Delta\vec{g} = \frac{\Delta\vec{G}}{\Delta V} = \frac{1}{c}(\vec{S}_{out} - \vec{S}_{in})\frac{\Delta A\cdot\Delta t}{\Delta A\cdot c\Delta t} = \frac{1}{c^2}(\vec{S}_{out} - \vec{S}_{in})$$

The momentum density (动量密度) of EMW: $\boxed{\vec{g} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2}(\vec{E} \times \vec{H})}$

Light Pressure

The pressure: For reflectivity 100% $P = \frac{2}{c}|\vec{S}_{in}| = \frac{2}{c}EH$

For reflectivity 0% for Black Body $P = \frac{1}{c}|\vec{S}_{in}| = \frac{1}{c}EH$

39-8 The Doppler (多普勒效应) effect for light wave

1. Sound wave, observer fixed, source moving away

$$f = f_0\frac{1}{1 + u/v}$$

2. Sound wave, source fixed, observer moving away

$$f = f_0(1 - u/v)$$

3. Light wave, source and observer separating (红移现象)

$$f = f_0\frac{1 - u/c}{\sqrt{1 - u^2/c^2}} = f_0\sqrt{\frac{1 - u/c}{1 + u/c}}$$

➢ Transverse (横向)Doppler effect:

At $\theta = \pi/2$, the light traveling perpendicular to the relative motion of the frames,

$$f = f_0\sqrt{1 - u^2/c^2}$$

purely relativistic effect, it leads to red-shift of the light.

$$f = f_0\frac{\sqrt{1 - u^2/c^2}}{1 + \frac{u}{c}\cos\theta}$$

where f_0 frequency measured in the frame the source is fixed.

➢ Longitudinal (纵向)Doppler effect:

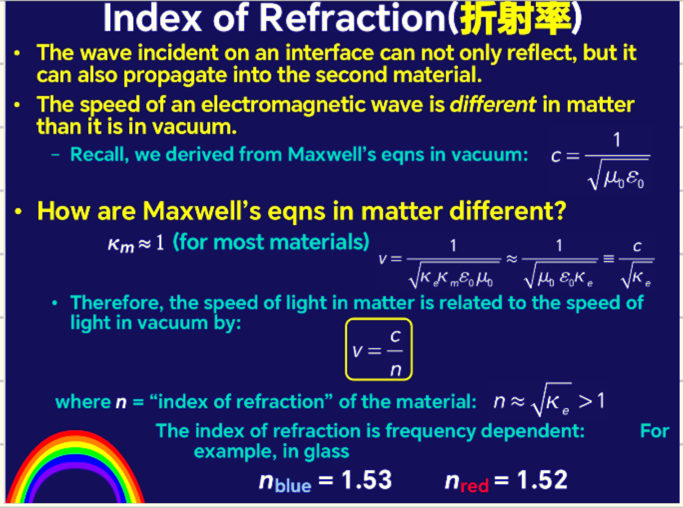
(a) the source approaching ($\theta = \pi$)

$$f = f_0\sqrt{\frac{1 + u/c}{1 - u/c}}$$

(b) the source leaving ($\theta = 0$)

$$f = f_0\sqrt{\frac{1 - u/c}{1 + u/c}}$$

·几何光学



Index of Refraction(折射率)

- The wave incident on an interface can not only reflect, but it can also propagate into the second material.
- The speed of an electromagnetic wave is *different* in matter than it is in vacuum.
  - Recall, we derived from Maxwell's eqns in vacuum: $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$
- How are Maxwell's eqns in matter different?

$\kappa_m \approx 1$ (for most materials) $v = \frac{1}{\sqrt{\kappa_e \kappa_m \varepsilon_0 \mu_0}} \approx \frac{1}{\sqrt{\mu_0 \varepsilon_0 \kappa_e}} \equiv \frac{c}{\sqrt{\kappa_e}}$

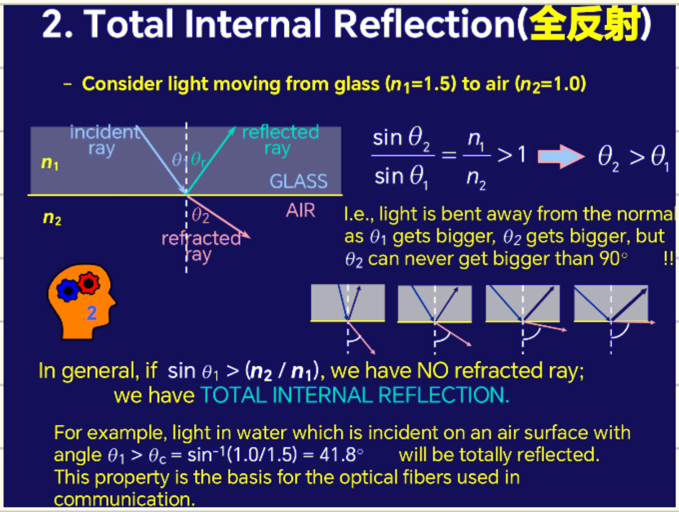
- Therefore, the speed of light in matter is related to the speed of light in vacuum by:

$$\boxed{v = \frac{c}{n}}$$

where n = "index of refraction" of the material: $n \approx \sqrt{\kappa_e} > 1$

The index of refraction is frequency dependent: For example, in glass

$n_{\text{blue}} = 1.53$ $n_{\text{red}} = 1.52$



2. Total Internal Reflection(全反射)

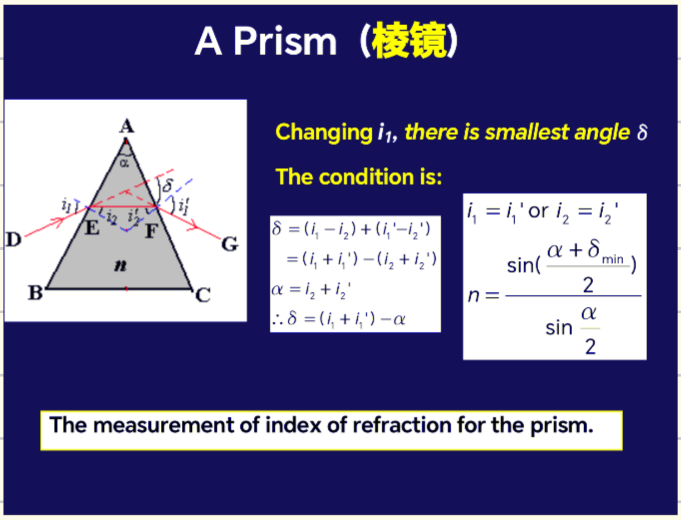
- Consider light moving from glass (n_1=1.5) to air (n_2=1.0)

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2} > 1 \Rightarrow \theta_2 > \theta_1$$

I.e., light is bent away from the normal as θ_1 gets bigger, θ_2 gets bigger, but θ_2 can never get bigger than 90° !!

In general, if $\sin\theta_1 > (n_2 / n_1)$, we have NO refracted ray; we have TOTAL INTERNAL REFLECTION.

For example, light in water which is incident on an air surface with angle $\theta_1 > \theta_c = \sin^{-1}(1.0/1.5) = 41.8°$ will be totally reflected. This property is the basis for the optical fibers used in communication.



A Prism（棱镜)

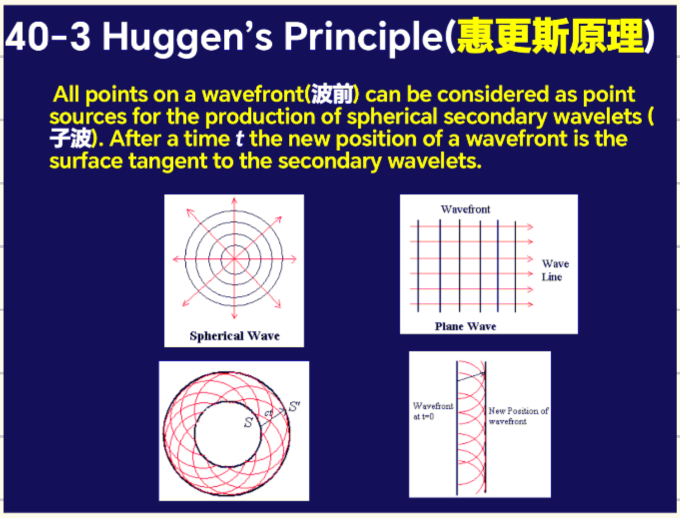
Changing i_1, there is smallest angle δ

The condition is:

$$\delta = (i_1 - i_2) + (i_1' - i_2') = (i_1 + i_1') - (i_2 + i_2')$$
$$\alpha = i_2 + i_2'$$
$$\therefore \delta = (i_1 + i_1') - \alpha$$

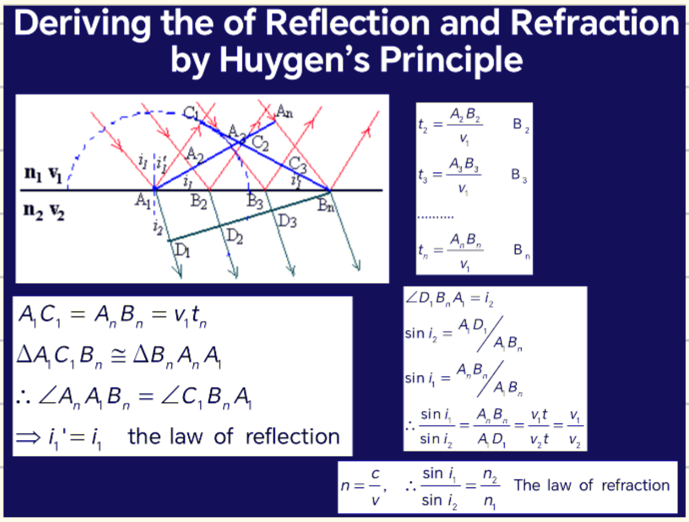
$$i_1 = i_1' \text{ or } i_2 = i_2'$$
$$n = \frac{\sin(\frac{\alpha + \delta_{\min}}{2})}{\sin\frac{\alpha}{2}}$$

The measurement of index of refraction for the prism.



40-3 Huggen's Principle(惠更斯原理)

All points on a wavefront(波前) can be considered as point sources for the production of spherical secondary wavelets (子波). After a time *t* the new position of a wavefront is the surface tangent to the secondary wavelets.



Deriving the of Reflection and Refraction by Huygen's Principle

$t_2 = \frac{A_2B_2}{v_1}$ B_2

$t_3 = \frac{A_3B_3}{v_1}$ B_3

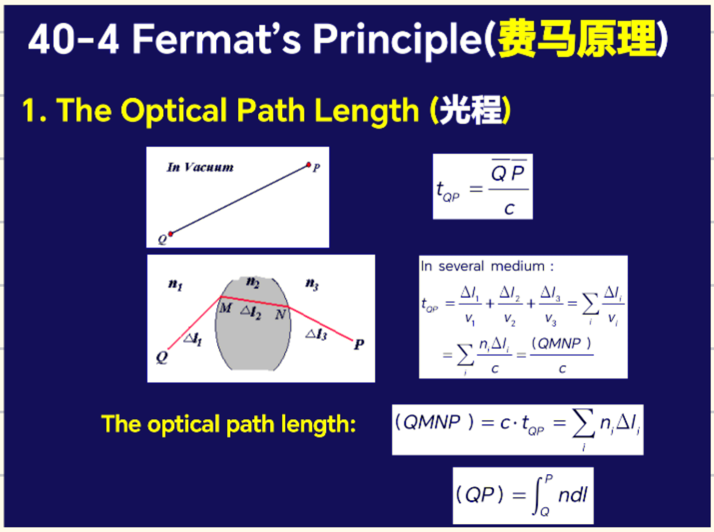
....

$t_n = \frac{A_nB_n}{v_1}$ B_n

$$A_1C_1 = A_nB_n = v_1 t_n$$
$$\Delta A_1C_1B_n \cong \Delta B_nA_nA_1$$
$$\therefore \angle A_nA_1B_n = \angle C_1B_nA_1$$
$$\Rightarrow i_1' = i_1 \quad \text{the law of reflection}$$

$$\angle D_1B_nA_1 = i_2$$
$$\sin i_2 = {A_1D_1}/{A_1B_n}$$
$$\sin i_1 = {A_nB_n}/{A_1B_n}$$
$$\therefore \frac{\sin i_1}{\sin i_2} = \frac{A_nB_n}{A_1D_1} = \frac{v_1 t}{v_2 t} = \frac{v_1}{v_2}$$

$$n = \frac{c}{v}, \quad \therefore \frac{\sin i_1}{\sin i_2} = \frac{n_2}{n_1} \quad \text{The law of refraction}$$



40-4 Fermat's Principle(费马原理)

1. The Optical Path Length (光程)

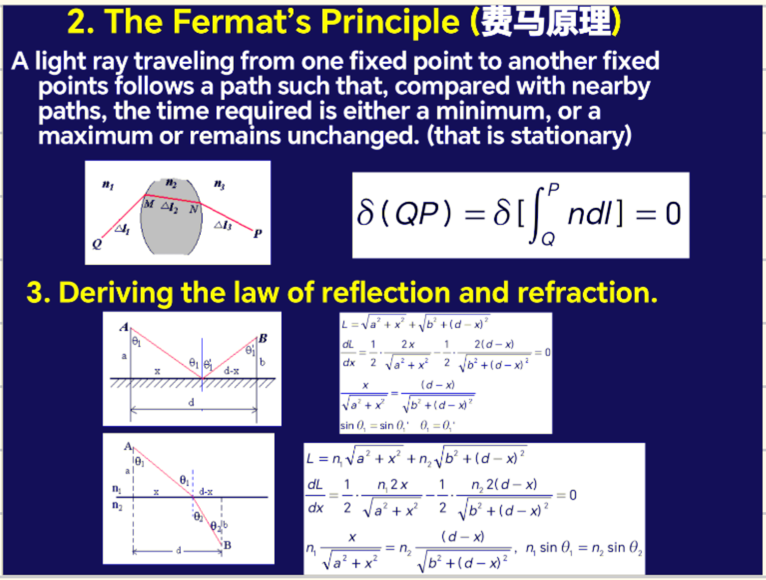
$$t_{QP} = \frac{\overline{Q}\,\overline{P}}{c}$$

In several medium :

$$t_{QP} = \frac{\Delta l_1}{v_1} + \frac{\Delta l_2}{v_2} + \frac{\Delta l_3}{v_3} = \sum_i \frac{\Delta l_i}{v_i} = \sum_i \frac{n_i \Delta l_i}{c} = \frac{(QMNP)}{c}$$

The optical path length: $(QMNP) = c \cdot t_{QP} = \sum_i n_i \Delta l_i$

$$(QP) = \int_Q^P n dl$$



2. The Fermat's Principle (费马原理)

A light ray traveling from one fixed point to another fixed points follows a path such that, compared with nearby paths, the time required is either a minimum, or a maximum or remains unchanged. (that is stationary)

$$\delta(QP) = \delta[\int_Q^P n dl] = 0$$

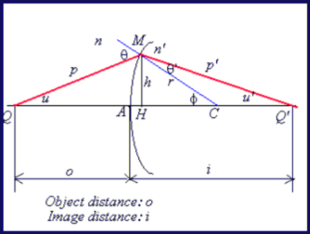
3. Deriving the law of reflection and refraction.

$$L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d-x)^2}$$
$$\frac{dL}{dx} = \frac{1}{2} \cdot \frac{2x}{\sqrt{a^2 + x^2}} - \frac{1}{2} \cdot \frac{2(d-x)}{\sqrt{b^2 + (d-x)^2}} = 0$$
$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{(d-x)}{\sqrt{b^2 + (d-x)^2}}$$
$$\sin\theta_1 = \sin\theta_1' \quad \theta_1 = \theta_1'$$

$$L = n_1\sqrt{a^2 + x^2} + n_2\sqrt{b^2 + (d-x)^2}$$
$$\frac{dL}{dx} = \frac{1}{2} \cdot \frac{n_1 2x}{\sqrt{a^2 + x^2}} - \frac{1}{2} \cdot \frac{n_2 2(d-x)}{\sqrt{b^2 + (d-x)^2}} = 0$$
$$n_1 \frac{x}{\sqrt{a^2 + x^2}} = n_2 \frac{(d-x)}{\sqrt{b^2 + (d-x)^2}}, \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

40-6 Image Formation by Spherical Mirror
(球面镜成像)

1. Refraction on a spherical surface:



$$\begin{cases} \dfrac{p}{\sin\phi} = \dfrac{o+r}{\sin\theta} = \dfrac{r}{\sin u} \\ \dfrac{p'}{\sin\phi} = \dfrac{i-r}{\sin\theta'} = \dfrac{r}{\sin u'} \end{cases}$$

$$n\sin\theta = n'\sin\theta'$$

$$\theta - u = \theta' + u' = \phi$$

$$\begin{cases} \dfrac{p}{o+r} = \dfrac{\sin\phi}{\sin\theta} \\ \dfrac{p'}{i-r} = \dfrac{\sin\phi}{\sin\theta'} \end{cases} \qquad n\sin\theta = n'\sin\theta'$$

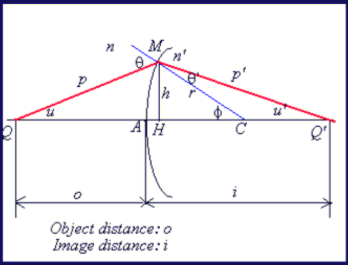
$$\therefore \frac{p}{n(o+r)} = \frac{p'}{n'(i-r)}$$

$$\begin{cases} p^2 = (o+r)^2 + r^2 - 2r(o+r)\cos\phi = o^2 + 4r(o+r)\sin^2\dfrac{\phi}{2} \\ p'^2 = (i-r)^2 + r^2 + 2r(i-r)\cos\phi = i^2 - 4r(i-r)\sin^2\dfrac{\phi}{2} \end{cases}$$

$$\cos\phi = 1 - 2\sin^2\frac{\phi}{2}$$

$$\Rightarrow \frac{o^2}{n^2(o+r)^2} - \frac{i^2}{n'^2(i-r)^2} = -4r\sin^2\frac{\phi}{2}\left[\frac{1}{n^2(o+r)} + \frac{1}{n'^2(i-r)}\right]$$

Cont.



$$\frac{o^2}{n^2(o+r)^2} - \frac{i^2}{n'^2(i-r)^2} = -4r\sin^2\frac{\phi}{2}\left[\frac{1}{n^2(o+r)} + \frac{1}{n'^2(i-r)}\right]$$

This result indicates that object Q point can not be imaged into Q' point through a spherical surface.

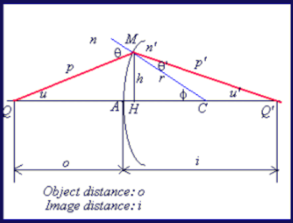
There are only two cases that **Q** point can be imaged into one point **Q'**:

$$\begin{cases} \dfrac{o^2}{n^2(o+r)^2} - \dfrac{i^2}{n'^2(i-r)^2} = 0 \\ \dfrac{1}{n^2(o+r)} + \dfrac{1}{n'^2(i-r)} = 0 \end{cases}$$

o, and *i* are determined at same time. For one spherical surface, there is only one group points. (齐明点)

Another case is for paraxial rays (傍轴近似) $h^2 << o^2, i^2, r^2$

2. Image Formation Equation (成像公式)



$$\frac{o^2}{n^2(o+r)^2} - \frac{i^2}{n'^2(i-r)^2} = -4r\sin^2\frac{\phi}{2}\left[\frac{1}{n^2(o+r)} + \frac{1}{n'^2(i-r)}\right]$$

For paraxial rays(傍轴光线):

$$u^2, u'^2, \phi^2 << 1, \quad \Rightarrow \theta^2, \theta'^2 << 1$$

$$\Rightarrow \sin^2\frac{\phi}{2} \approx \left(\frac{\phi}{2}\right)^2 \to 0$$

$$\frac{o^2}{n^2(o+r)^2} = \frac{i^2}{n'^2(i-r)^2}$$

$$\frac{n(o+r)}{o} = \frac{n'(i-r)}{i}$$

$$\frac{n'}{i} + \frac{n}{o} = \frac{n'-n}{r}$$

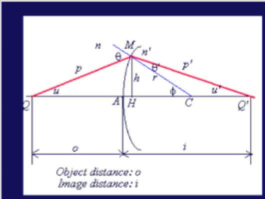
For any Q point (object distance *o*, there is image Q' point (image distance *i*).

The first focal point: 第一焦点 $i \to \infty, \; o = f = \dfrac{n}{n'-n} r$

The second focal point: 第二焦点 $o \to \infty, \; i = f' = \dfrac{n'}{n'-n} r$

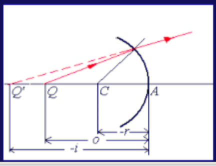
$$\boxed{\frac{f}{f'} = \frac{n}{n'}, \qquad \frac{f}{o} + \frac{f'}{i} = 1}$$

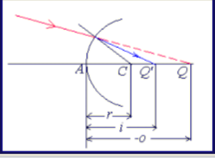
3. Sign Conventions
(符号约定)



If we suggest that the incident light ray from left to right.

(1) If the *Q* point is at the left of A point (实物) *o>0*
If the *Q* point is at the right of A point (虚物) *o<0*
(2) If the *Q'* point is at the left of A point (虚像) *i<0*
If the *Q'* point is at the right of A point (实像) *i>0*
(3) If the *C* point (球心) is at the left of A point (凹)， *r<0*
If the *C* point (球心) is at the right of A point (凸)，r>0

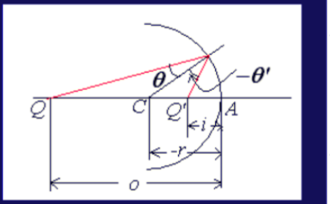




虚物成实像

For the reflection at the surface of a mirror
(球面反射成像)

(2') If the *Q'* point is at the left of A point (实像), *i >0*
If the *Q'* point is at the right of A point (虚像), *i <0*



$$n\sin\theta = n'\sin\theta'$$

if $\theta > 0$, then $\theta' < 0$

$$n = -n'$$

$$\frac{n'}{i} + \frac{n}{o} = \frac{n'-n}{r}$$

$$\frac{f}{o} + \frac{f'}{i} = 1;$$

$$\frac{(-\frac{r}{2})}{o} + \frac{\frac{r}{2}}{(-i)} = 1$$

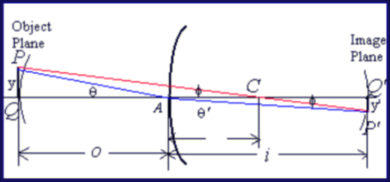
$$\Rightarrow \frac{1}{o} + \frac{1}{i} = -\frac{2}{r}$$

$$f = \frac{n}{n'-n} r = -\frac{r}{2}$$

$$f' = \frac{n'}{n'-n} r = \frac{1}{2} r$$

$$\frac{1}{o} + \frac{1}{i} = -\frac{2}{r}$$

4. The image formation of paraxial object point and lateral magnification
(傍轴物点成像和横向放大率)



Paraxial Ray: $y^2, y'^2 << o^2, i^2, r^2$

Sign convention: (4) If *P (or P')* is above the light axis, *y (or y')>0*
If *P (or P')* is below the light axis, *y (or y')<0*

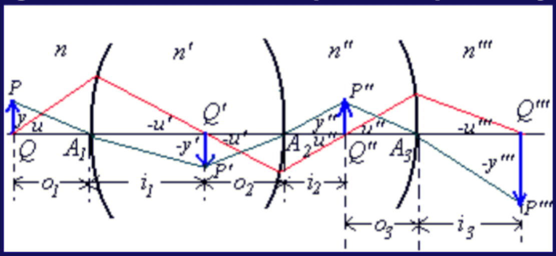
Lateral Magnification: $m = \dfrac{\text{Lateral Size of Image}}{\text{Lateral Size of Object}} = \dfrac{y'}{y}$

Paraxial Ray: $n\theta \approx n'\theta', \quad y \approx o\cdot\theta, \quad -y' = i\cdot\theta'$

$$\therefore m = \frac{y'}{y} = -\frac{i\theta'}{o\theta} = -\frac{n\cdot i}{n'\cdot o}$$

For the reflection: $m = -\dfrac{i}{o}$

5. Image Formation of Compound Optical System



$$\begin{cases}\dfrac{n'}{i_1}+\dfrac{n}{o_1}=\dfrac{n'-n}{r_1}\\ \dfrac{n''}{i_2}+\dfrac{n'}{o_2}=\dfrac{n''-n'}{r_2}\\ \dfrac{n'''}{i_3}+\dfrac{n''}{o_3}=\dfrac{n'''-n''}{r_3}\end{cases}$$

$$\frac{f_1'}{i_1}+\frac{f_1}{o_1}=1 \qquad \frac{f_2'}{i_2}+\frac{f_2}{o_2}=1 \qquad \frac{f_3'}{i_3}+\frac{f_3}{o_3}=1$$

$$m_1=-\frac{ni_1}{n'o_1} \qquad m_2=-\frac{n'i_2}{n''o_2} \qquad m_3=-\frac{n''i_3}{n'''o_3}$$

$$u\approx\frac{h}{QA_1}=\frac{h}{o_1} \qquad -u'=\frac{h}{A_1Q'}=\frac{h}{i_1} \qquad \therefore\frac{u}{u'}=-\frac{i_1}{o_1}$$

$$m=-\frac{ni}{n'o}=\frac{y'}{y}=\frac{nu}{n'u'}$$

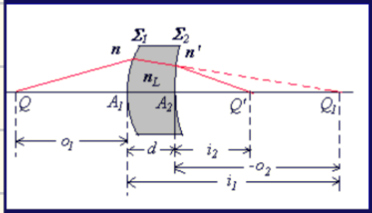
Lagrange-Helmholtz Law

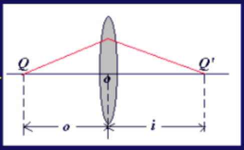
$$ynu=y'n'u'=y''n''u''=\ldots\ldots\ldots$$

40-7 Thin Lens (薄透镜)

In most refraction situation there is more than one refracting surface.

1. The formula of focal length(焦距) *f*.





Thin lens, *d* is very small, A_1, A_2 become one point: o: optical center (光心)

$$\begin{cases}\dfrac{f_1'}{i_1}+\dfrac{f_1}{o_1}=1\\ \dfrac{f_2'}{i_2}+\dfrac{f_2}{o_2}=1\end{cases}$$

$$f_1=\frac{n}{n_L-n}r_1,\quad f_1'=\frac{n_L}{n_L-n}r_1 \qquad f_2=\frac{n_L}{n'-n_L}r_2,\quad f_2'=\frac{n'}{n'-n_L}r_2$$

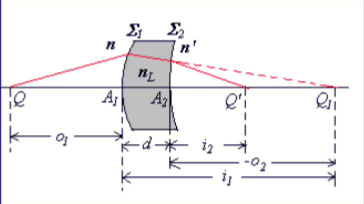
$$-o_2=i_1-d,\ o_2=d-i_1$$

$$o=\overline{Q}\,\overline{O}\approx o_1 \qquad i=\overline{O}\,\overline{Q}'\approx i_2 \qquad -o_2\approx i_1$$

$$\begin{cases}\dfrac{f_1'f_2}{i_1}+\dfrac{f_1f_2}{o_1}=f_2\\ \dfrac{f_1'f_2'}{i_2}+\dfrac{f_1'f_2}{(-i_1)}=f_1'\end{cases} \rightarrow \quad \frac{f_1'f_2'}{i_2}+\frac{f_1f_2}{o_1}=f_1'+f_2 \qquad \frac{f_1'f_2'}{i}+\frac{f_1f_2}{o}=f_1'+f_2$$

$$\boxed{\frac{f'}{i}+\frac{f}{o}=1}$$

Lens maker's Equation (磨镜者公式)



$$f_1=\frac{n}{n_L-n}r_1,\quad f_1'=\frac{n_L}{n_L-n}r_1 \qquad f_2=\frac{n_L}{n'-n_L}r_2,\quad f_2'=\frac{n'}{n'-n_L}r_2$$

$$\frac{f_1'f_2'}{i_2}+\frac{f_1f_2}{o_1}=f_1'+f_2 \qquad \frac{f_1'f_2'}{i}+\frac{f_1f_2}{o}=f_1'+f_2$$

$$\frac{f'}{i}+\frac{f}{o}=1$$

$$f'=\frac{f_1'f_2'}{f_1'+f_2}=\frac{\frac{n_L}{n_L-n}\cdot\frac{n'}{n'-n_L}r_1r_2}{\frac{n_L}{n_L-n}r_1+\frac{n_L}{n'-n_L}r_2}=\frac{n'}{\frac{n_L-n}{r_1}+\frac{n'-n_L}{r_2}}$$

$$f=\frac{f_1f_2}{f_1'+f_2}=\frac{\frac{n}{n_L-n}\cdot\frac{n_L}{n'-n_L}r_1r_2}{\frac{n_L}{n_L-n}r_1+\frac{n_L}{n'-n_L}r_2}=\frac{n}{\frac{n_L-n}{r_1}+\frac{n'-n_L}{r_2}}$$

$$\therefore\frac{f'}{f}=\frac{n'}{n}$$

If n=n'=1

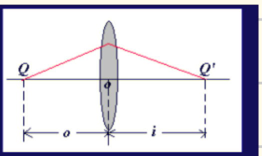
$$f=f'=\frac{1}{(n_L-1)(\frac{1}{r_1}-\frac{1}{r_2})}$$

- ***If f>0, f'>0 Converging lens* (凸透镜)**

  A lens that is thicker at the center than at the edges.

- ***If f<0, f'<0 diverging lens* (凹透镜)**

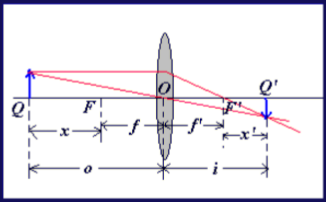
  A lens that is thicker at the edges than at the center.



$$\frac{f'}{i}+\frac{f}{o}=1$$

$$f=f'=\frac{1}{(n_L-1)(\frac{1}{r_1}-\frac{1}{r_2})}$$

2. The Image Formation Formula



$$\boxed{\frac{f'}{i}+\frac{f}{o}=1}$$

If n=n', f=f'

Gauss' Form

$$\frac{1}{i}+\frac{1}{o}=\frac{1}{f}$$

Sign Convention:

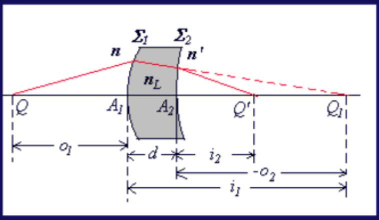
(6) If *Q* is at the left of *F* point, *x*>0
If *Q* is at the right of *F* point, *x*<0
(7) If *Q'* is at the left of *F'* point, *x'*<0
If *Q'* is at the right of *F'* point, *x'*>0

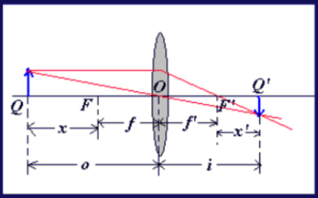
$$o=f+x \qquad i=f'+x' \qquad \frac{1}{f+x}+\frac{1}{f'+x'}=\frac{1}{f}$$

Newton's Form:

$$xx'=f^2=ff'$$

Lateral Magnification (横向放大倍数)





$$o_1=o \qquad -o_2=i_1 \qquad i_2=i$$

$$m_1=-\frac{ni_1}{n_Lo_1},\quad m_2=-\frac{n_Li_2}{n'o_2}$$

$$m=m_1m_2=\frac{ni_1}{n_Lo_1}\cdot\frac{n_Li_2}{n'o_2}=\frac{ni_1}{n_Lo_1}\cdot\frac{n_Li}{n'(-i_1)}=-\frac{ni}{n'o}=-\frac{fi}{f'o}$$

$$m=-\frac{f}{x}=-\frac{x'}{f'}$$

For example:

Diopter, D (屈光度)

$$P=\frac{1}{f(m)}$$

$$f=-50cm=-0.5m,\ P=\frac{1}{-0.5}=-2.00D$$

200度